Some results in Noncommutative Geometry and (Noncommutative) Topology: Semifinite spectral triples associated with some self-coverings, the 2-adic ring C^*-algebra of the integers, and the oriented Thompson group.

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December 19, 2016

Contents

Introduction **2**

1 Some facts about C*-algebras and noncommutative geometry **7**

1.1 C*-algebras and the commutative Gelfand-Naimark Theorem 7
1.2 Inductive limits 10
1.3 Discrete crossed products 13
1.4 Uniformly hyperfinite algebra 14
1.5 Cuntz algebras and the 2-adic ring C*-algebra 15
1.6 Clifford algebras 18
1.7 Definition and examples of spectral triples and semifinite spectral triples 19
1.8 Compact quantum metric spaces 22
1.9 Functions of positive type 25

2 Noncommutative solenoidal spaces from self-coverings **27**

2.1 Noncommutative coverings w.r.t. a finite abelian group 27
2.1.1 Spectral decomposition 27
2.1.2 Noncommutative coverings 28
2.1.3 Representations 30
2.1.4 Finite regular coverings 31
2.2 Self-coverings of tori 32
2.2.1 The C*-algebra and its spectral triple 32
2.2.2 The covering 32
2.2.3 Spectral triples on covering spaces of $\mathbb{T}^p$ 34
2.2.4 The inductive limit spectral triple 36
2.3 Self-coverings of rational rotation algebras 39
2.3.1 Coverings of noncommutative tori 39
2.3.2 The C*- algebra, a spectral triple and the self-covering 40
2.3.3 Spectral triples on noncommutative covering spaces of A_θ 42
2.3.4 The inductive limit spectral triple 44
2.4 Self-coverings of crossed products 45
2.4.1 The C*-algebra, its spectral triple and the self-covering 45
2.4.2 Spectral triples on covering spaces of $\mathcal{Z} \rtimes_\rho \mathbb{Z}^p$ 49
2.4.3 The inductive limit spectral triple 51
2.5 Self-coverings of UHF-algebras 52

2.5.1 The C*-algebra, the spectral triple and an endomorphism 52
2.5.2 Spectral triples on covering spaces of UHF-algebras 53
2.6 Inductive limits and the weak*-topology of their state spaces 55
2.7 Appendix: Some results in noncommutative integration theory 58

3 The inner structure of $\mathcal{Q}_2$ and its automorphism group 62
3.1 Some preliminary results 62
3.1.1 Some extensible endomorphisms 62
3.1.2 The gauge-invariant subalgebra 63
3.1.3 The canonical representation 65
3.2 Structure results 66
3.2.1 Two maximal abelian subalgebras 66
3.2.2 The relative commutant of the generating isometry 71
3.3 Extending endomorphisms of the Cuntz algebra 75
3.3.1 Uniqueness of the extensions 75
3.3.2 Extensible Bogoljubov automorphisms 78
3.4 Outer automorphisms 81
3.4.1 Gauge automorphisms and the flip-flop 81
3.4.2 A general result 83
3.5 Notable endomorphisms and automorphism classes 84
3.5.1 Endomorphisms and automorphisms α such that $\alpha(S_2) = S_2$ . . . 84
3.5.2 Automorphisms α such that $\alpha(U) = U$ 86
3.5.3 Automorphisms α such that $\alpha(U) = zU$ 89
3.6 Appendix: The functional equation $f(z^2) = f(z)^2$ on the torus 92

4 The HOMFLYPT polynomial and the oriented Thompson group 94
4.1 Preliminaries: some definitions and examples 94
4.2 Main result: the Homflypt polonomial as a positive type function 98

Introduction

Since the beginning of my Ph.D. I have been mainly interested in three topics, all related to Operator Algebras, namely Noncommutative Geometry, Noncommutative Topology, and Knot Theory. Before going into the details of my work during these three years I would like to spend a few words about some of these topics.

Noncommutative Topology and Geometry have their origin in Gelfand-Naimark Theorem which establishes a duality between algebras and topological spaces. Given a compact Hausdorff space X one can consider the C^*-algebra of continuous functions $C(X)$. On the other hand, given a unital, commutative C^*-algebra $\mathcal{A}$, it turns out that it is isometrically isomorphic to $C(\widehat{\mathcal{A}})$, where $\widehat{\mathcal{A}}$ is the spectrum of the algebra. Actually, Gelfand-Naimark Theorem says more, in particular that there exists an explicit contravariant functor between the category $\mathcal{T}$ whose objects are compact Hausdorff spaces and morphisms are continuous maps and the category $\mathcal{C}$ whose objects are commutative, unital C^*-algebras, with morphisms given by unital $*$- homomorphisms

$$\begin{aligned}
&\mathcal{T} \to \mathcal{C} \\
&X \mapsto C(X) \\
&\phi \mapsto \hat{\phi}
\end{aligned}$$

With this result in mind, noncommutative C^*-algebras can be thought as continuous functions on noncommutative spaces. This idea was pursued further by Alain Connes introducing Noncommutative Geometry. Motivated by the fact that the Dirac Operators of contains many pieces of information of a manifold, the notion of spectral triple was introduced.

Noncommutative selfcoverings and inductive limits

Consider a family of self-coverings, namely a sequence of homeomorphic spaces connected by covering maps

$$X_0 \overset{\pi_0}{\twoheadrightarrow} X_1 \overset{\pi_1}{\twoheadrightarrow} X_2 \overset{\pi_2}{\twoheadrightarrow} X_3 \ldots$$

In the commutative case, by Gelfand-Naimark duality the spaces become algebras and the covering maps become injective morphisms. Therefore, there is an associated sequence of embedding of algebras

$$\mathcal{A}_0 \overset{\alpha}{\hookrightarrow} \mathcal{A}_1 \overset{\alpha}{\hookrightarrow} \mathcal{A}_2 \overset{\alpha}{\hookrightarrow} \mathcal{A}_3 \ldots$$

Motivated by this fact, when the algebras $\mathcal{A}_i$ are no longer abelian, but are still pairwise isomorphic, we consider the above sequence and think of the algebras as a sequence of noncommutative self-coverings. This sequence yields an inductive limit $\varinjlim \mathcal{A}_i$. Suppose that the first algebra is endowed with a spectral triple. We considered the problem of extending the spectral triple to whole inductive limit. We studied many cases, for example the case of self-coverings of p-tori, of noncommutative tori, some crossed products with respect to $\mathbb{Z}^p$ and UHF algebras. In those cases it is possible to construct natural spectral triples on each C^*-algebra $\mathcal{A}_n$ and the sequence of the Dirac operators is shown to converge in a suitable sense, to a triple on the inductive limit. In all the cases considered the spectral triples for the inductive limit C^*-algebra are semifinite.

Our leading idea is to produce geometries on each (noncommutative) space $\mathcal{A}_n$ which are locally isomorphic to the geometry on the original noncommutative space $\mathcal{A}$. In other words, the covering projections should be local isometries or, phrased in terms of Rieffel's theory of compact quantum metric spaces (cf. [Con94, Rie99]), that the noncommutative metrics given by the Lip-norms associated with the Dirac operators via $L_n(a) = \|[D_n, a]\|$ should be compatible with the inductive maps, *i.e.* $L_{n+1}(\alpha(a)) = L_n(a)$, $a \in \mathcal{A}_n$. However, in one case, this property will be weakened to the existence of a finite limit for the sequences $L_{n+p}(\alpha^p(a))$, $a \in \mathcal{A}_n$.

There are two main consequences of the former assumption. The first one is that the noncommutative coverings are metrically larger and larger, so that their radii diverge to infinity, and *the inductive limit is topologically compact* (the C*-algebra has a unit) *but not totally bounded* (the metric on the state space does not induce the weak*-topology). The second one is that the spectrum of the Dirac operator becomes more and more dense in the real line, so that the resolvent of the limiting Dirac operator cannot be compact, being indeed τ-compact w.r.t. a suitable trace, and thus producing a *semifinite spectral triple on the inductive limit.*

The 2-adic ring C*-algebra of the integers

The 2-adic ring C^*-algebra $\mathcal{Q}_2$ is the object of one of the chapters of this thesis, however before defining it, we begin with introducing the Cuntz algebras. For any natural number n greater than 2, the Cuntz algebra $\mathcal{O}_n$ was introduced in [Cun77] as the universal C*-algebra generated by n isometries $\{S_i\}_{i=1}^n$ such that

$$\sum_{i=1}^n S_i S_i^* = 1 \qquad S_i^* S_j = \delta_{i,j} \ .$$

This was the first concrete example of a separable infinite simple C*-algebra and, since then, Cuntz algebras have received a great deal of attention. Later, these algebras were generalized in many different ways. For example, a few years ago Joachim Cuntz [Cun08] introduced the C^*-algebra $\mathcal{Q}_\mathbb{N}$ associated with the $ax + b$-semigroup over the natural numbers and, some years later, Larsen and Li [LL12] considered its 2-adic version. The 2-adic ring C^*-algebra $\mathcal{Q}_2$ can be described as the universal C^*-algebra generated by a unitary U and an isometry S_2 such that

$$S_2U = U^2S_2 \quad \text{and} \quad S_2S_2^* + US_2S_2^*U^* = 1$$

We mention that this algebra appeared before, for example in [Hir02], but it is in the above-mentioned work that it was studied systematically for the first time. We recall that among other things $\mathcal{Q}_2$ is a nuclear, purely infinite, simple C^*-algebra.

One of our strongest motivations in studying $\mathcal{Q}_2$ is the fact that it contains $\mathcal{O}_2$ and we wanted to understand the relation between these two algebras. The inclusion can be realized in the following way. Consider the map taking S_1 to US_2 and S_2 to S_2, it extends by universality to a homomorphism from $\mathcal{O}_2$ to $\mathcal{Q}_2$. This morphism is injective by the simplicity of $\mathcal{O}_2$ and so we may think $\mathcal{O}_2$ as a subalgebra of $\mathcal{Q}_2$.

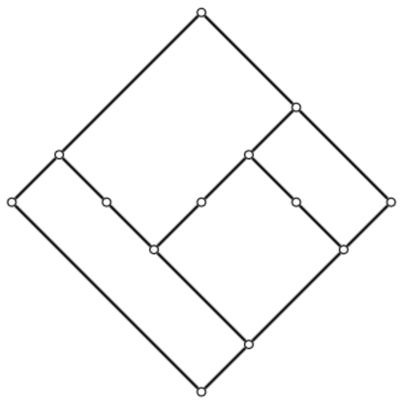
The main objectives of Chapter 3 are to describe the structure of $\mathcal{Q}_2$ and study its automorphisms/endomorphism. In the particular, the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$ naturally leads to problem of the determining what morphisms of the smaller algebra extends to the bigger algebra. We briefly recall that to any unitary $x \in \mathcal{U}(\mathcal{O}_2)$ there is an associated endomorphism of $\mathcal{O}_2$ that maps S_1 and S_2 to xS_1 and xS_2, respectively. It was pointed out by Takesaki that all the endomorphisms have this form. It turns out that not all endomorphisms extends. For instance, by looking at Bogoljubov automorphisms many examples of non-extensible automorphisms can be found. Determining precisely which are extensible is one of the aims of Chapter 3.

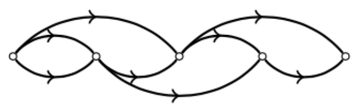
Besides the problem of extensible automorphisms, other families of endomorphisms are studied. For example those fixing the unitary generator U. This problem turned out to be related to the structure of the C^*-subalgebra generated by U. In particular, we found out that $C^*(U)$ is a maximal subalgebra of $\mathcal{Q}_2$ and this allowed us to give a complete description of these endomorphisms, actually all automatically automorphisms of type quasi-free in the sense of Dykema-Shlyakhtenko and Zacharias, see [Dyk01, Zac00].

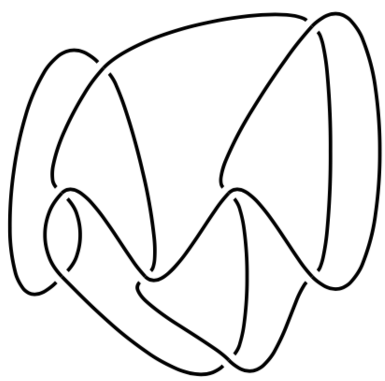
We end this section with a few words concerning outer automorphisms. In the case of $\mathcal{O}_2$ the group $\mathrm{Out}(\mathcal{O}_2)$ is a large group. In fact, by using Bogoljubov automorphisms one can embed almost all locally compact groups. For $\mathcal{Q}_2$ a similar statement is not known whether it is true or not since not all Bogoljubov automorphisms are extensible. It seems that $\mathrm{Out}(\mathcal{Q}_2)$ is not as large as $\mathrm{Out}(\mathcal{O}_2)$. Nevertheless, the outer automorphism group is shown to be at least uncountable and non-abelian. The latter again relies upon the fact that $C^*(U)$ is maximal abelian in $\mathcal{Q}_2$.

The Thompson groups and Knot Theory

For the third theme, I have been working on the relation between the Thompson groups and Knot Theory. This relation stems from the recent research activity of Vaughan F. R. Jones who discovered an unexpected relation between knots and the Thompson groups F and T, [Jon14] and [Jon16]. Actually he defined two different procedures to associate a link to each element of the Thompson groups. Both procedures are related to the possible values of the variable of the Kauffman bracket $\langle\cdot\rangle(A)$, one for the real values of the variable and one for A being a root of unity. We refer to the first construction with the expression *real procedure*, whereas for the second we use the expression *complex procedure*. These procedures rely on the fact that elements of the Thompson groups can be described in terms of pairs of rooted planar binary trees with the same number of leaves. Given an element $g \in F$, the picture below illustrates how to construct $\mathcal{L}(g)$ from such a pair of trees according to the real construction







These procedures had oriented versions that allow to give a canonical orientation to the associated link. However, these constructions are well defined only on the oriented Thompson groups $\vec{F}$ and $\vec{T}$, both introduced in [Jon14]. We mention that Jones proved that the Thompson groups are as good as the Braid groups for describing links. In fact, he proved an Alexander-type Theorem which says that any link can be obtained with the complex procedure. So it is possible to replace the familiar Braid groups in the description of knots and links, thus opening a new line of research on the interplay between F and low-dimensional topology.

By using Jones's work it is possible to evaluate suitably renomalized graph and link invarants and possibly obtain a positive type function on the group. This in turn would yields a unitary representations of the group.

Outline of the thesis

The First Chapter deals with the main prerequisites for the topics covered in this thesis. The aim is to make it self-contained. In particular the fundamental definition and results concerning Noncommutative Topology are introduced. The Gelfand-Naimark Theorem and fundamental properties of C*-algebras are presented. Some examples of C*-algebras that will be relevant in the sequel are described, namely the Cuntz Algebra, the 2-adic ring C*-algebra, the UHF-algebra. The definitions of Spectral Triples and compact quantum metric spaces are also discussed. At the end of the Chapter the notion of positive type function on a discrete group is discussed.

The Second Chapter deals with the notion of covering in the framework of Noncommutative Geometry. In particular, some examples of noncommutative self-coverings are discussed, and spectral triples on the base space are extended to spectral triples on the inductive family of coverings. These families of spectral triples yield a (semifinite) spectral triple for the inductive limit algebra. Some of the self-coverings described were given by the inclusion of the fixed point algebra in a C*-algebra acted upon by a finite abelian group. The results discussed in this chapter are contained in the recent paper [AGI] (with Daniele Guido and Tommaso Isola both affiliated to Università di Roma Tor Vergata).

The third Chapter deals with some results concerning the 2-adic ring C*-algebra $\mathcal{Q}_2$. First of all, some features of the inner structure of this algebra are obtained and then many consequences concerning endomorphisms are drawn. For instance the triviality of the relative commutant $C^*(S_2)' \cap \mathcal{Q}_2$ yields a rigidity property the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$, namely endomorphisms of $\mathcal{Q}_2$ that restrict to the identity on $\mathcal{O}_2$ are actually the identity on the whole $\mathcal{Q}_2$. Furthermore, the two subalgebras $\mathcal{D}_2$ and $C^*(U)$ are shown to be maximal abelian. By using this property of the latter subalgebra, the general form of endomorphisms fixing U is determined. Actually the semigroup of the endomorphisms fixing U turned out to be a maximal abelian subgroup of $\mathrm{Aut}(\mathcal{Q}_2)$ topologically isomorphic with $C(\mathbb{T}, \mathbb{T})$. These results are contained in [ACR16] (with Roberto Conti affiliated to Sapienza Università di Roma) and Stefano Rossi affiliated to Università di Roma Tor Vergata).

In the Fourth Chapter it is shown that the Homflypt polynomial yields a positive type function on the oriented Thompson group $\vec{F}$. This result rely on the so-called *real procedure* introduced by V. Jones. This is a self-contained description of the result obtained in [AC16b] (with Roberto Conti). We mention that the *complex procedure* is dealt with different techniques in the very recent paper [ACJ16] (with Roberto Conti and Vaughan F. R. Jones affiliated to the Vanderbilt University).

Chapter 1

Some facts about C*-algebras and noncommutative geometry

1.1 C*-algebras and the commutative Gelfand-Naimark Theorem

In this chapter we recall the definition and some basic properties of C^*-algebras. The reader is referred to [Dav96, Chapter 1], [Ped12] or [Dix77] for further details and proofs of the stated results.

An associative algebra is a $\mathbb{C}$-vector space $\mathcal{A}$ with a bilinear map

$$\begin{array}{rcl} \mathcal{A} \times \mathcal{A} & \to & \mathcal{A} \\ (a, b) & \to & ab \end{array}$$

such that

$$a(bc) = (ab)c \quad \forall a, b, c \in A.$$

An associative algebra $\mathcal{A}$ is called associative Banach algebra if it is a complete normed algebra and the norm is such that

$$\|xy\| \leq \|x\| \|y\| \qquad \forall x, y \in \mathcal{A} \, .$$

The former inequality ensures that the multiplication operation is continuous.

Actually we will always work with algebras endowed with an additional structure: an involution. An involution is a map $* : \mathcal{A} \to \mathcal{A}$ such that

1. $(x^*)^* = x$,
2. $(x + y)^* = x^* + y^*$,
3. $(\lambda x)^* = \overline{\lambda} x^*$,
4. $(xy)^* = y^* x^*$,

where $x, y \in \mathcal{A}$, $\lambda \in \mathbb{C}$. The pair $(\mathcal{A}, *)$ is called involutive algebra.

Given $x \in \mathcal{A}$, the element x^* is called adjoint of x. An element is called self-adjoint if $x^* = x$, normal if $x^*x = xx^*$. A projection is an idempotent, self-adjoint element. An element is unitary if $x^*x = xx^* = 1$. An isometry is an element such that $x^*x = 1$.

We observe that every element $x \in \mathcal{A}$ can be written as the sum of two self-adjoint elements, namely

$$\begin{aligned} x &= x_1 + ix_2 \\ x_1 &:= \frac{1}{2}(x + x^*), \\ x_2 &:= \frac{1}{2i}(x - x^*). \end{aligned}$$

It can actually be shown that any element is the linear combination of four unitary elements (cf. [Ped12, Lemma 3.2.21, p. 97]).

When $\mathcal{A}$ is Banach algebra with an involution and it holds $||a^*a|| = ||a||^2$ for all $a \in \mathcal{A}$, we say that $\mathcal{A}$ is a C^*-algebra.

Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, by a morphism we mean a map $\phi : \mathcal{A} \to \mathcal{B}$ such that

1. $\phi(x + y) = \phi(x) + \phi(y)$
2. $\phi(\lambda x) = \lambda\phi(x)$
3. $\phi(xy) = \phi(x)\phi(y)$
4. $\phi(x^*) = \phi(x)^*$

where $x, y \in \mathcal{A}$, $\lambda \in \mathbb{C}$.

The following result ensures that morphisms are automatically continuous (with respect to the norm topology).

Proposition 1.1.1. *(*[Dix77, Proposition 1.3.7, p. 9; Proposition 1.8.1, p. 20]*) Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism between two C^*-algebras. Then φ is continuous. Moreover, if φ is injective, then φ is an isometry.*

Proposition 1.1.2. *Let $\mathcal{A}$ be a non-unital C^*-algebra, then there exists a unital C^*-algebra $\tilde{\mathcal{A}}$ containing $\mathcal{A}$ as a maximal ideal of co-dimension* 1.

Proof. Consider the embedding $\phi : \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{A})$ (the Banach algebra of bounded operators on $\mathcal{A}$) given by the left regular representation $\phi(a)(x) = ax$. The map ϕ is an isometric morphism. In fact

$$\begin{aligned} ||\phi(a)|| &\leq ||a||, \\ ||a||^2 = ||aa^*|| = ||\phi(a)a^*|| &\leq ||\phi(a)||||a^*||. \end{aligned}$$

Set $\tilde{\mathcal{A}} = \rho(\mathcal{A}) + 1\mathbb{C}$, where 1 is the identity operator in $\mathcal{B}(\mathcal{A})$. We observe that $\tilde{\mathcal{A}}$ is complete since $\mathcal{A}$ and $\tilde{\mathcal{A}}/\mathcal{A} = \mathbb{C}$ are complete. We define an involution on $\tilde{\mathcal{A}}$ as $(\phi(a) + \lambda)^* := \phi(a^*) + \overline{\lambda}$. We check that $\tilde{\mathcal{A}}$ is a C^*-algebra. For all $\epsilon > 0$ there exists an $y \in \mathcal{A}$, $\|y\| \leq 1$, such that

$$\begin{aligned} \|\phi(x) + \alpha 1\|^2 &\leq \epsilon + \|(x+\alpha)y\|^2 \\ &= \epsilon + \|y^*(x^* + \overline{\alpha})(x+\alpha)y\| \\ &\leq \epsilon + \|(x^* + \overline{\alpha})(x+\alpha)y\| \\ &\leq \epsilon + \|(\phi(x)+\alpha)^*(\phi(x)+\alpha)\|, \end{aligned}$$

whence $\|\phi(x) + \alpha 1\|^2 \leq \|(\phi(x)+\alpha)^*(\phi(x)+\alpha)\|$. Since the inequality $\|(\phi(x) + \alpha)^*(\phi(x)+\alpha)\| \leq \|\phi(x) + \alpha 1\|^2$ is clear, we are done. □

Let $\mathcal{A}$ be a C^*-algebra. The spectrum of an element is defined as

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} | \lambda 1 - a \text{ is not invertible in } \tilde{\mathcal{A}}\} .$$

It can be shown that the spectrum of an element does not depend on the choice of the C^*-algebra containing the element, see [Dix77, Proposition 1.3.10, p. 10].

An example of commutative C^*-algebra is $C(X)$, where X is a compact Hausdorff space with the norm being

$$\|f\| := \sup_{x \in X} |f(x)| \quad f \in C(X) ,$$

and the involution given by $f^*(z) := \overline{f(z)}$. It is a remarkable fact that all commutative are of this form (up to $*$-isometric isomorphism). We will properly state this result known as Gelfand-Naimark Theorem.

Suppose that $\mathcal{A}$ is a commutative C^*-algebra. A character is a homorphism $\tau : \mathcal{A} \to \mathbb{C}$. The set of characters is denoted by $\widehat{\mathcal{A}}$ and is called the spectrum of the algebra. It is possible to verify that the set of characters is a closed subset of the unit ball of $\mathcal{A}^*$ with respect to weak-* topology. The Gelfand transform $\Gamma : \mathcal{A} \to C(\widehat{\mathcal{A}})$ is defined as $\Gamma(a)(\tau) := \tau(a)$ with $\tau \in \widehat{\mathcal{A}}$, $a \in \mathcal{A}$. Given a continuous map $\phi : X \to Y$ between compact Hausdorff space there is a naturally induced morphism $\hat{\phi} : C(Y) \to C(X)$ given by $\hat{\phi}(f)(y) := f(\phi(y))$.

Theorem 1.1.3. *(*[Bla06, II.2.2.6, p.61]*) Let $\mathcal{T}$ be the category whose objects are compact Hausdorff spaces and morphisms are continuous maps, and denote by $\mathcal{C}$ the category whose objects are commutative, unital C^*-algebras, with morphisms given by unital $*$-homomorphisms. Then the correspondence*

$$\begin{aligned} &\mathcal{T} \to \mathcal{C} \\ &X \mapsto C(X) \\ &\phi \mapsto \hat{\phi} \end{aligned}$$

is a contravariant category equivalence.

The former result is one of the starting point of noncommutative topology, because from this we may think that noncommutative C^*-algebras corresponds to "noncommutative spaces".

The following table form a basic "dictionary" for the compact case [1]

Topological framework	**C*-algebraic framework**
compact space	unital C^*-algebra
continuous map	morphism
measure	positive functional
connected space	projectionless C^*-algebra
second countable	separable C^*-algebra

1.2 Inductive limits

This section is devoted to recall the definition and the fundamental properties of inductive limits. The concept of inductive limit is a generalization of the notion of union. We refer for example to [WO94, Appendix L, p. 298] for further details.

We will start with the case of groups, algebras, and $*$-algebras. We will refer to them with the term "algebraic objects". Then, we will consider the C^*-algebraic case.

Let I be a directed set, i.e. it is a set endowed with a reflexive, transitive, and binary relation $\leq$ with the following property: for any i, $j \in I$ there exists an element $k \in I$ such that $i \leq k$ and $j \leq k$. Now let $\{A_i\}_{i\in I}$ be a family of algebraic objects. In order to define the inductive limit we need a directed system. A directed system is a collection of morphisms $\Phi_{ij} : A_j \to A_i$ for all $i \leq j$ satisfying the compatibility condition $\Phi_{ij} = \Phi_{ik} \circ \Phi_{kj}$ whenever $j < k < i$.

With these notations, the algebraic inductive limit is an object $A_\infty = \varinjlim A_i$ together with a family of morphisms $\Phi_i : A_i \to A_\infty$, called canonical morphisms, such that the following diagram commutes

$$\begin{array}{ccc} A_j & \xrightarrow{\Phi_j} & A_\infty \\ \Big\downarrow{\scriptstyle \Phi_{ij}} & \nearrow{\scriptstyle \Phi_i} & \\ A_i, & & \end{array}$$

and such that $A_\infty = \cup \Phi_i(A_i)$.

An explicit construction of the inductive limit is the following

$$A_\infty \quad := \quad \left\{ (a_i) \in \prod_{i\in I} A_i \mid \exists i_0 \in \ I : \forall \ i > i_0 \Rightarrow a_i = \Phi_{ii_0}(a_{i_0}) \right\} \Big/ \sim \tag{1.2.1}$$

where $(b_i) \sim (c_i)$ if exists an index i_k such that $b_i = c_i$ when $i > i_k$. The set A_∞ can be endowed with operations according to the category we are working in. For example,

[1] This small dictionary is taken from [WO94, p. 24].

if we are working in the framework of groups (i.e. all A_i are groups, all Φ_{ij} are group homomorphisms), then we may define the product componentwise. The same idea may be applied to algebras and $*$-algebras in order to obtain an object in the desired category. We define the canonical morphisms as follows

$$\Phi_j(x) := (a_i), \text{ where } \begin{cases} a_i = \Phi_{ij}(x) & \text{if } j \leq i \\ a_i = 0 & \text{otherwise} \end{cases} .$$

The definition of algebraic inductive limit is universal, in the sense that it satisfies a universal property.

Proposition 1.2.1. *(Universal property of the inductive limit) Let N be an object with a collection of morphisms $\Psi_i : A_i \to N$ such that $\Psi_i \circ \Phi_{ij} = \Psi_j$ whenever $j < i$. Then it exists a unique morphism $\Xi : A_\infty \to N$ such that the following diagram commutes*

$$\begin{tikzcd} A_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"] \arrow[dr, "\Psi_j"] & A_\infty \arrow[d, "\Xi"] \\ A_i. \arrow[r, "\Psi_i"] \arrow[ur, "\Phi_i"] & N. \end{tikzcd}$$

The following property will be useful whenever one wants to prove that the morphism Ξ, define above, is bijective.

Proposition 1.2.2. *Suppose that for all $j < i$ the following*

$$\begin{tikzcd} A_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"] \arrow[dr, "\Psi_j"] & A_\infty \arrow[d, "\Xi"] \\ A_i. \arrow[r, "\Psi_i"] \arrow[ur, "\Phi_i"] & N. \end{tikzcd}$$

If $N = \cup \Psi_i(A_i)$, then Ξ is surjective.
If $\forall i$ Ψ_i is injective, then Ξ is injective.

Proof. First of all we take care of surjectivity. Let $A_\infty = \varinjlim A_i = \cup \Phi_i(A_i)$. By using the commutativity of the diagram we obtain the following chain of equalities

$$N = \cup_i \Psi_i(A_i) = \cup_i \Xi \circ \Phi_i(A_i) = \Xi \varinjlim A_i.$$

Now, let's consider the part of the claim corresponding to injectivity. Let $x, y \in A_\infty$ and suppose $\Xi(x) = \Xi(y)$. By definition of inductive limit we have that $x = \Phi_i(a_i)$ and $y = \Phi_j(a_j)$ for some $a_i \in A_i$ e $a_j \in A_j$. Since I is directed set, there exists an index $k \in I$ such that $k > i$, $k > j$. Let $x_k := \Phi_{ki}(a_i)$, $y_k := \Phi_{kj}(a_j)$. By definition we have that $x = \Phi_k(x_k)$ and $y = \Phi_k(y_k)$. Then

$$\Psi_k(x_k) = \Xi\Phi_k(x_k) = \Xi(x) = \Xi(y) = \Xi\Phi_k(y_k) = \Psi_k(y),$$

since the morphisms Ψ_k are injective, we get that $x = y$. □

Immediate consequences of the former Proposition are the following results.

Corollary 1.2.3. *With the above notations, suppose that for all $i, j \in I$ the objects A_i are all isomorphic to an object M. Moreover, suppose that the following diagram commutes*

$$\begin{tikzcd} A_j \arrow[r, "\Phi_{ij}"] \arrow[d, "\sim"] & A_i \\ M \arrow[ur, "\sim"'] & \end{tikzcd}$$

Then M is isomorphic to the inductive limits of the directed system $\{A_j, \Phi_{ij}\}$.

Corollary 1.2.4. *Denote by A_∞ and B_∞ the inductive limits of the directed systems $\{A_n, \Phi_{mn}\}$ and $\{B_n, \Psi_{mn}\}$, respectively. Suppose that the there exists a sequence of indices*

$$n_1 \le m_1 < n_2 \le m_2 < \dots$$

and for each k a morphism $\alpha_k : A_{n_k} \to B_{m_k}$ such that the following diagram commute

$$\begin{tikzcd} A_{n_1} \arrow[r] \arrow[dr, "\alpha_1"] & \cdots \arrow[r] & A_{n_2} \arrow[r] \arrow[dr, "\alpha_1"] & \cdots \arrow[r] & \cdots \arrow[r] & A_\infty \\ B_{n_1} \arrow[r] & B_{m_1} \arrow[r] \arrow[ur, "\beta_1"] & \cdots \arrow[r] & B_{m_2} \arrow[r] \arrow[ur, "\beta_1"] & \cdots \arrow[r] & B_\infty \end{tikzcd}$$

Then the inductive limits A_∞ and B_∞ are isomorphic.

Let A an algebra and consider the matrix algebras $M_n(A)$. There are natural inclusions $i_{mn} : M_n(A) \to M_m(A)$, $i(a) = \text{diag}(a, 0_{m-n})$ if $n \le m$. The algebraic inductive limit will be denoted by $M_\infty(A)$.

Now we are ready to give the definition of inductive limit for C^*-algebras. With the former notations, suppose that A_i is a collection of C^*-algebras and that Φ_{ij} are $*$-homorphisms. By the former discussion we may construct the algebraic inductive limit A_∞ . We define the following C^*-seminorm on A_∞. Let $x = \Phi_j(a_j)$, then $\alpha(x) := \limsup_j \|\Phi_{ij}(a_{ij})\|$. Consider the ideal $N := \{x \in A_\infty | \alpha(x) = 0\}$ and consider A_∞ / N. Now the former seminorm becomes a C^*-norm. After completion we obtain an algebra that we call the inductive limit of C^*-Algebras. This algebra will be denoted with the same symbol used in the algebraic framework. This is the corresponding universal property.

Proposition 1.2.5. *(Universal property of the C^*-inductive limit) Let M be another C^*-algebra and let $\Psi_i : A_i \to M$ be a family of morphism such that $\Psi_i \circ \Phi_{ij} = \Psi_j$ when*

$j \le i$. Then there exists a unique morphism $\Xi : A_\infty \to N$ such that

$$\begin{tikzcd} A_j \arrow[r, "\Phi_j"] \arrow[d, "\Phi_{ij}"'] \arrow[dr, "\Psi_j"] & A_\infty \arrow[d, "\Xi"] \\ A_i . \arrow[r, "\Psi_i"] \arrow[ur, "\Phi_i"'] & M. \end{tikzcd}$$

1.3 Crossed products with respect to an action of a discrete group

The aim of this short section is to introduce crossed products with respect to a discrete countable group. Since the objective is give a self-contained treatment of an example considered in Chapter 2 we decided not to consider the most general definition of crossed product. This section is inspired by [Dav96, Chapter VIII, p. 216].

A triple $(\mathcal{A}, G, \alpha)$ where $\mathcal{A}$ is a unital C^*-algebra, $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a group homomorphism, will be called a C^*-dynamical system. A pair (π, U), where $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation, $U : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation such that

$$U_g \pi(A) U_g^* = \pi(\alpha_g(A)) \qquad \forall A \in \mathcal{A}, g \in G \ ,$$

is called covariant representation.

Consider the vector space $C_c(G, \mathcal{A})$ of finitely supported functions on G with values in $\mathcal{A}$. Elements of $C_c(G, \mathcal{A})$ will be of the form $\sum_{g \in G} a_g \delta_g$ (with finitely non-zero elements a_g) where $\delta_g(x) := \delta_{g,x}$ is the Kronecker delta. This vector space can be endowed with structure of an algebra by setting $\delta_g a \delta_{g^{-1}} = \alpha_g(a)$ for $g \in G$, $a \in \mathcal{A}$. The product of two finitely supported function $h = \sum_{g \in G} a_g \delta_g$ and $k = \sum_{g \in G} b_g \delta_g$ is thus given by

$$h \cdot k = \sum_{g,i \in G} a_g \alpha_g(b_{g^{-1}i}) \delta_i \ .$$

The involution is determined by setting $s^* = s^{-1}$ for all $s \in G$. This implies that

$$\left(\sum_{g \in G} a_g \delta_g \right)^* = \sum_{g \in G} \alpha_g(a^*_{g^{-1}}) \delta_g \ .$$

We observe that there exists a bijective correspondence between $*$-representations of $C_c(G, \mathcal{A})$ and covariant representations. In fact, given a covariant representation one may define $\sigma : C_c(G, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ as $\sigma(f) = \sum_{g \in G} \pi(a_g) U_g$. Conversely, given a $*$-representation $\sigma : C_c(G, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ one can obtain a covariant representation by setting $\pi(a) = \sigma(a \cdot e)$ and $U_g = \sigma(g)$ where $e \in G$ is the neutral element.

The crossed product algebra $\mathcal{A} \rtimes_\alpha G$ is obtained as the enveloping C^*-algebra of $C_c(G, \mathcal{A})$, namely one defines the following C^*-norm

$$\|f\| := \sup_\sigma \|\sigma(f)\|$$

where σ runs over all the *-representations of $C_c(G,\mathcal{A})$. We observe that the supremum is always bounded by

$$\|f\|_1 = \sum_g \|a_g\| \ .$$

We stress that there exist $*$-representations. In fact, given a *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ one can consider the tensor product with left regular representation of G. In particular we get the following covariant representation $(\tilde{\pi}, \tilde{U})$ on the Hilbert space $\ell_2(G,\mathcal{A})$

$$(\tilde{\pi}(a)x)(g) := \pi(\alpha_g^{-1}(a))(x(g))$$
$$(\tilde{U}_t x)(g) := x(t^{-1}g) \ .$$

with $x \in \ell_2(G,\mathcal{A})$. We mention that in general the representations that we defined using the left regular representation of G are not sufficient to determine the norm of $\mathcal{A} \rtimes_\alpha G$. These representations determine the so-called reduced crossed product. The two constructions coincide when G is amenable. Since we will only be interested in the case $G \cong \mathbb{Z}^p$ these two definitions will coincide.

We observe that when $\mathcal{A} = \mathbb{C}$, and G acts trivially on $\mathcal{A}$, we obtain the group C^*-algebra.

1.4 Uniformly hyperfinite algebra

In this section we recall the definition of the Uniformly Hyper-Finite algebra, also known as UHF algebra. This algebra will appear in the present chapter as a subalgebra of the Cuntz algebras and the 2-adic C^*-algebra, and will be studied in Chapter 2. This C^*-algebra was studied and classified for the first time by James G. Glimm in his Ph.D. thesis (published in [Gli60]). We refer to [RLL00, Section 7.4] for further details.

A Uniformly Hyper-Finite algebra is a C^*-algebra isomorphic to the inductive limit of a sequence of matrix algebras $\{M_{k_i}(\mathbb{C})\}_i$ with unital connecting homomorphisms $\{\phi_{i+1,i}\}$.

Remark 1.4.1. We observe that there exists a unital homorphism $\phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ if and only if $m = dn$. In fact, let $\{e_{ij}^{(n)}\}_{i,j=1}^n$ be a family of matrix units. Given a homomorphism ϕ, since all the projections $e_{ii}^{(n)}$ are unitarily equivalent, then also $\phi(e_{ii}^{(n)})$ are all unitarily equivalent. This means that

$$m = \mathrm{tr}(1_m) = \sum_{i=1}^n \mathrm{tr}(\phi(e_{ii}^{(n)})) = n\,\mathrm{tr}(\phi(e_{11}^{(n)}))$$

where $\mathrm{tr}(\cdot)$ denotes the un-normalized trace. Since $\mathrm{tr}(\phi(e_{11}^{(n)})) \in \mathbb{N}$ (this can be seen by observing that $\mathrm{sp}(\phi(e_{11}^{(n)})) \subset \mathrm{sp}(e_{11}^{(n)}) = \{0,1\}$) we get the claim. If on the other hand $m = dn$ we may consider the embedding $\phi(x) := (x,\dots,x)$ for all $x \in M_n(\mathbb{C})$.

By the former remark it follows that it is not restrictive to suppose that k_i divides k_{i+1}.

Consider the set $\{p_1, p_2, \ldots\}$ of all positive prime numbers listed in increasing order. A supernatural number is a sequence $n = \{n_j\}_{j\geq 1}$ with each $n_i \in \mathbb{N} \cup \{\infty\}$. We may denote n as the formal infinite product

$$n = \prod_{j\geq 1} p_i^{n_i} .$$

The operation being $n \cdot m := \{n_j\}_{j\geq 1} + \{m_j\}_{j\geq 1} = \{n_j + m_j\}_{j\geq 1}$.

To any supernatural number n we may associate the subgroup $Q(n)$ of the group $(\mathbb{Q}, +)$ defined as the subgroup of $\mathbb{Q}$ whose elements are x/y with x being an integer and $y = \prod_{\geq 1} p_i^{m_i}$ for some $m_i \leq n_i$ and $m_i \neq 0$ only for finitely many i. When the UHF is already an inductive limit of matrix algebras it may be seen that

$$Q(n) = \cup_{i\geq 1} k_i^{-1}\mathbb{Z} .$$

In this case, it can be shown that the K_0-group is given by $Q(n)$. Since any UHF has K_0-group of the form $Q(n)$, we will call n the supernatural number associated to the UHF algebra.

Actually for every supernatural number there exists a UHF algebra whose associated supernatural number is n. In fact, let n be a supernatural number and set

$$k_j = \prod_{i=1}^{j} p_i^{\min\{j, n_i\}}$$

By construction, k_j divides k_{j+1} for every j. Set $\phi_j : M_{k_j}(\mathbb{C}) \to M_{k_{j+1}}(\mathbb{C})$ by $\phi_j(x) := (x, \ldots, x)$. By construction the inductive limit C^*-algebra has associated supernatural number n.

Moreover, we have the following classification result.

Theorem 1.4.2. *(*[RLL00, Theorem 7.4.5, p. 128]*) Let $\mathcal{A}$ and $\mathcal{A}'$ be UHF algebras associated with supernatural numbers n and m, respectively. Then the following are equivalent*

- *$\mathcal{A}$ and $\mathcal{A}'$ are isomorphic;*
- *the supernatural numbers n and n' are equal;*
- *the groups $(K_0(\mathcal{A}), [1_{\mathcal{A}}])$ and $(K_0(\mathcal{A}'), [1_{\mathcal{A}'}])$ are isomorphic (i.e. there exists a group isomorphims $\alpha : K_0(\mathcal{A}) \to K_0(\mathcal{A}')$ such that $\alpha([1_{\mathcal{A}}]) = [1_{\mathcal{A}'}]$).*

1.5 Cuntz algebras and the 2-adic ring C*-algebra

In this section we recall the definition of the Cuntz algebras and of the 2-adic ring C^*-algebra. For the first topic we refer to [Dav96], in particular to Chapter 5, and to Cuntz' paper [Cun77], whereas for the second we refer to [LL12].

For each $n \in \mathbb{N}$ the Cuntz algebra $\mathcal{O}_n$ is defined as the universal C^*-algebra generated by n isometries $S_1, \ldots, S_n$ such that

$$S_i^* S_j = \delta_{i,j} \qquad \sum_{i=1}^{n} S_i S_i^* = 1 \ .$$

The above above conditions means that the isometries have orthogonal ranges and that the orthogonal sum ranges is the whole space.

The algebra $\mathcal{O}_2$ is quite remarkable since many C^*-algebras embeds into it. For precise statement of this property the interested reader is referred to a paper of Kirchberg and Phillips ([KP00]). However, as we will see some algebras have explicit embeddings. First of all, we will fix the notation. Set

$$W_n^k = \begin{cases} \varnothing & \text{for } k = 0 \\ \{1, \cdots, n\}^k & \text{for } k \geq 1 \end{cases}$$
$$W_n = \bigcup_{k=0}^{\infty} W_n^k \ .$$

We will call elements of W_n multi-indices. An element $\alpha \in W_n^k$ is said to have length k and we denote by $\ell(\alpha)$ (or $|\alpha|$) this number.

First of all we exhibit an embedding of the algebra $M_n(\mathbb{C})$ in $\mathcal{O}_n$.

$$\begin{aligned} M_n(\mathbb{C}) &\to \mathcal{O}_n \\ e_{ij} &\mapsto S_i S_j^* \end{aligned}$$

Actually, the above morphism extends to an embeddings of the UHF algebra of type 2^∞

$$\begin{aligned} \mathrm{UHF}(n^\infty) &\to \mathcal{O}_n \\ e_{i_1,j_1} \otimes \ldots \otimes e_{i_k,j_k} &\mapsto S_\alpha S_\beta^* \end{aligned}$$

where $\alpha = (i_1, \ldots, i_k)$, $\beta = (j_1, \ldots, j_k)$, $S_\alpha = S_{i_1} \cdots S_{i_k}$ and $S_\alpha = S_{j_1} \cdots S_{j_k}$. We will denote by $\mathcal{F}_n$ the C^*-subalgebra of $\mathcal{O}_n$ isomorphic to the UHF algebra of type 2^∞ that we considered above, namely

$$\mathcal{F}_n := \overline{\mathrm{span}}\{S_\alpha S_\beta^*, \ \ell(\alpha) = \ell(\beta)\} \ .$$

Another remarkable C^*-subalgebra is the diagonal subalgebra

$$\mathcal{D}_n := \overline{\mathrm{span}}\{S_\alpha S_\alpha^*\} \ .$$

It can be proved that $\mathcal{D}_n$ is a maximal abelian subalgebra of $\mathcal{O}_n$.

An interesting fact is that the Cuntz algebras are simple. In particular, this means that any endomorphisms is either injective or the trivial endomorphism (i.e. the one that maps any element to 0).

The fact that two Cuntz algebras $\mathcal{O}_m$ and $\mathcal{O}_n$ are not isomorphic, unless $m = n$, follows from the fact that the K-groups are

$$K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1} \quad K_1(\mathcal{O}_n) = 0$$

From now on, we by endomorphism we will mean unital endomorphism. We will denote by $\mathrm{End}(\mathcal{O}_n)$ the semigroup of unital endomorphisms. Given a unitary element $u \in \mathcal{U}(\mathcal{O}_n)$ it is easy to see that the $\lambda_u(S_i) := uS_i$ defines a endomorphism. It is a remarkable fact found by Takesaki (as credited in Cuntz' paper [Cun80]) that the map

$$\begin{aligned} \mathcal{U}(\mathcal{O}_n) &\to \mathrm{End}(\mathcal{O}_n) \\ u &\mapsto \lambda_u \end{aligned}$$

is bijective. This map we will be called the Cuntz-Takesaki correspondence. The automorphisms associated with the unitaries of $\mathcal{F}_n^1 \cong M_n(\mathbb{C})$ will be called the Bogoljubov automorphisms.

Now we introduce the 2-adic ring C^*-algebra. This algebra was introduced in [LL12]. This is the universal C^*-algebra $\mathcal{Q}_2$ generated by a unitary U and an isometry S_2 such that

$$S_2U = U^2S_2 \quad \text{and} \quad S_2S_2^* + US_2S_2^*U^* = 1 \ .$$

We notice that $\mathcal{Q}_2$ contains a distinguished copy of $\mathcal{O}_2$. In fact, if we set $S_1 := US_2$ we can easily see that S_1, S_2 satisfy the relations of the $\mathcal{O}_2$. In particular, $\mathcal{Q}_2$ contains $\mathcal{D}_2$ and $\mathcal{F}_2$.

It can be proved that also $\mathcal{Q}_2$ is simple and the above considerations apply.

Since it will be very useful in the sequel, we introduce the so called canonical representation. Denote by $\{e_k\}_{k\in\mathbb{Z}}$ the canonical basis of $\ell_2(\mathbb{Z})$ and define

$$\pi : \mathcal{Q}_2 \to \mathcal{B}(\ell_2(\mathbb{Z}))$$

as $\pi(S_2)e_k := e_{2k}$, $\pi(U)e_k := e_{k+1}$. We observe that $\pi(S_2S_2^*)\ell_2(\mathbb{Z}) = \overline{\mathrm{span}}\{e_{2i}, i \in \mathbb{Z}\}$. This representation is faithful. We observe this representation restricts to a representation of $\mathcal{O}_2$. Larsen and Li discussed necessary and sufficient conditions under which one may extend a representation of $\mathcal{O}_2$ to $\mathcal{Q}_2$. In particular, a representation is extensible if and only if the unitary components of the Wold decomposition of the isometries are unitary equivalent.

1.6 Clifford algebras

The purpose of this short section is to recall the definition and the fundamental properties of Clifford algebras. For further details the interested reader is referred to [LM16] and [GBVF00].

Let V be a finite dimensional vector space over a field $\mathbb{K}$ (of characteristic $\neq 2$) and q be a symmetric bilinear form. The Clifford algebra $(Cl(V,q), i)$ may be defined as the quotient of the tensor algebra $\mathcal{T}(V) = \mathbb{K} \oplus \bigoplus_{k\geq 1} V^{\otimes k}$ with respect to the ideal $\mathcal{I}(V)$ generated by the elements $v \otimes w + w \otimes v - 2q(v,w)$ with $v, w \in V$. We observe that the natural projection $\pi : \mathcal{T}(V) \to Cl(V,q)$ restricts to an injective map $i := \pi \restriction_V : V \to Cl(V,q)$.
The Clifford algebra satisfies the following universal property.

Proposition 1.6.1. *Let A be a unital associative algebra and $\Phi : V \to A$ a linear map such that*

$$\Phi(v)\Phi(w) + \Phi(w)\Phi(v) = 2q(v,w)1_A \qquad v, w \in V.$$

Then, there exists a unique endomorphism $\tilde{\Phi} : Cl(V,q) \to A$ that extends Φ. Moreover, given another algebra $Cl'(V,q)$ together with a linear map $j : V \to Cl'(V,q)$ such that $j(v)v(w) + j(w)j(v) = q(v,w)$, then $Cl(V,q)$ and $Cl'(V,q)$ are isomorphic.

Proof. We begin by showing the existence of such an extension of the morphim. The universal property of the tensor product yield a morphism $\Phi_\otimes : \mathcal{T}(V) \to A$. Since $\mathcal{I}(V) \subset \ker \Phi_\otimes$, thus $\Phi_\otimes$ induces a morphism $\tilde{\Phi} : Cl(V,q) \to A$. The extension is actually unique. Indeed, given to maps $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ be two extensions. Then, they coincide on V, thus on $\mathcal{T}(V)$ and also on $Cl(V,q)$.

Now, we take care of the uniqueness of the Clifford algebra. Let $Cl'(V,q')$ be another algebra satisfying the former property. Consider the two inclusions $i : V \to Cl(V,q)$ and $j : V \to Cl'(V,q')$ such that $i(v)^2 = q(v)1$, $j(v)^2 = q'(v)1$. These maps extends to morphisms $\alpha : Cl(V,q) \to Cl'(V,q')$ and $\beta : Cl'(V,q') \to Cl(V,q)$, respectively. We have that $i = \beta \circ \alpha \circ i$, $j = \alpha \circ \beta \circ j$. By the uniqueness of the extension we have that $\alpha \circ \beta = id_{Cl(V,q)}$ and $\beta \circ \alpha = id_{Cl'(V,q)}$. Therefore, the morphisms β and α are actually one the inverse of the other. □

The former proposition tells us that up to isomorphism there exists a unique Clifford algebra associated with the pair (V,q).

A vector space E is called Clifford module if there exists an homorphism $c : Cl(V,q) \to \mathrm{End}(E)$. If E is endowed with an inner product $(\cdot,\cdot)_E$, the module E is said a unitary Clifford module if for all norm-one vectors $v \in V$ the operator $c(v) \in O(E, (\cdot,\cdot)_E)$.

The Clifford algebra $Cl(V,q)$ is $\mathbb{Z}_2-$graded ([LM16, p. 9]). In fact, consider the automorphism $\alpha : Cl(V,q) \to Cl(V,q)$ defined as $\alpha(v) = -v$ for all $v \in V$. This, automorphism is clearly an involution. Thus $Cl(v,q) = Cl(v,q)^0 \oplus Cl(v,q)^1$, with $Cl(V,q)^j = \{x \in Cl(V,q) \mid \alpha(x) = (-1)^j x\}$.

We denote by $\mathbb{C}l(V,q) := Cl(V,q) \otimes \mathbb{C}$ When $q(z_1, \ldots, z_n) = \sum_{i=1}^n z_i^2$, we will denote $\mathbb{C}l(\mathbb{R}^n, q)$ simply by $\mathbb{C}l(\mathbb{R}^n)$.

Theorem 1.6.2. *(*[Fri00, p. 13]*) We have that*

- *If* $n = 2m$, *then* $\mathbb{C}l(\mathbb{R}^n) \cong \mathrm{End}[(\mathbb{C}^2)^{\otimes k}] \cong \mathrm{End}[\mathbb{C}^{2^k}]$.
- *if* $n = 2m+1$, *then* $\mathbb{C}l(\mathbb{R}^n) \cong \big(\mathrm{End}[(\mathbb{C}^2)^{\otimes k}]\big) \oplus \big(\mathrm{End}[(\mathbb{C}^2)^{\otimes k}]\big) \cong \mathrm{End}[\mathbb{C}^{2^k}] \oplus \mathrm{End}[\mathbb{C}^{2^k}]$.

In particular $\dim(\mathbb{C}l(\mathbb{R}^n)) = 2^n$.

We mention that the above vector space $\mathbb{C}^{2^k}$, $n = 2k,\ 2k+1$, is referred to as the complex vector space of n-spinors, [Fri00, p. 14].

Now we define inductively a representation of the Clifford algebra $\mathbb{C}l(\mathbb{R}^n)$, see [GBVF00, p. 333]. Let $\{e_i\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. Here and in the following chapter we will denote the represented vectors of the canonical basis with $\{\epsilon_i^{(n)}\}_{i=1}^n$ For $n = 1$ set $\epsilon_1^{(1)} = 1$. For n odd

$$\epsilon_j^{(n)} := \begin{pmatrix} 0 & \epsilon_j^{(n-2)} \\ \epsilon_j^{(n-2)} & 0 \end{pmatrix}, \quad \epsilon_{n-1}^{(n)} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \epsilon_n^{(n)} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For n even just set $\epsilon_j^{(n)} := \epsilon_j^{(n)+1} :=$ for $j = 1, \ldots, n$. We observe that for $n = 3$ we get the so-called Pauli matrices ([GBVF00, p. 76]), namely

$$\epsilon_1^{(3)} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon_2^{(3)} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \epsilon_3^{(3)} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1.7 Definition and examples of spectral triples and semifinite spectral triples

In this section we introduce the definition of spectral triple (first the "classical" type I case), then the one for the semifinite setting, [Con94]. We will present some basic examples of spectral triples, some of which will yield examples of compact quantum metric spaces.

An odd spectral triple is triple $(\mathcal{A}, \mathcal{H}, D)$ with

1. $\mathcal{A}$ is a unital dense $*$-subalgebra of a C^*-algebra A;
2. $\mathcal{H}$ is Hilbert space;
3. A is faithfully represented on $\mathcal{H}$;
4. D is an (unbounded) self-adjoint operator acting on $\mathcal{H}$;
5. D has compact resolvent;
6. $[D, a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$.

If there exists a self-adjoint unitary operator Γ such that $\Gamma^2 = 1$ and $\Gamma D = -D\Gamma$, the spectral triple is said to be even.

Example 1.7.1. *(The circle)* The first example is a "commutative" example, the case of the circle. Consider the triple $(C^1(\mathbb{T}), L^2(\mathbb{T}), D = -i\partial_\theta)$ (cf. [Vár06, Example 3.2, p. 34]). We check the commutator condition. By using the product rule for the derivative we get

$$[D, f](g) = D(fg) - fD(g) = -i(f'g + fg') - fg' = -if'g$$

which is bounded since $f \in C^1(\mathbb{T})$. The condition on the resolvent follows from the fact that we have that there exists a basis of eigenvector $(\{e^{2\pi in\theta}\}_{n\in\mathbb{N}})$ with eigenvalues going to infinity as $n \to \infty$ (i.e. $D(e^{2\pi in\theta}) = -2\pi ine^{2\pi in\theta}$).

Example 1.7.2. *(The group C^*-algebra)* The second example is the basic example of spectral triple associated with the reduced group C^*-algebra of a countable group Γ. This example was introduced in Connes' paper [Con89]. We recall that a length function[2] on a group Γ is a function $\ell : \Gamma \to \mathbb{R}^*$ satisfying

1. $\ell(g) = 0$ if and only if $g = e$;
2. $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in \Gamma$;
3. $\ell(g^{-1}) = \ell(g)$ for all $g \in \Gamma$.

For example, for the group $(\mathbb{Z}^p, +)$ one can consider $\ell(g) := ||g||^2$, where $||\cdot||$ is the euclidean norm of $\mathbb{R}^p$.

The group reduced group C^*-algebra $C_r^*(\Gamma)$ is given by the closure of the linear span of the operators $\lambda(g) : \ell^2(\Gamma) \to \ell^2(\Gamma)$ defined as $\lambda(g)\delta_h = \delta_{g^{-1}h}$. So by definition the reduced group C^*-algebra $C_r^*(\Gamma)$ acts faithfully on $\ell^2(\Gamma)$. Consider the multiplication operator $D = M_\ell$. Set $\mathbb{C}[\Gamma] := \mathrm{span}\{\lambda(g), g \in \Gamma\}$. If $\#\, \ell^{-1}([0, c]) < \infty$ for all $c \in \mathbb{R}^+$, then $(\mathbb{C}[\Gamma], \ell^2(\Gamma), D)$ is a spectral triple. For example, let's check the condition on the bounded commutator (cf. [Con89, Lemma 5, p. 210]). We observe that it is enough to check that $[D, \lambda(g)]$ is a bounded operator. We have that

$$\begin{aligned}[D, \lambda(g)]\delta_h &= D\lambda(g)\delta_h - \lambda(g)D\delta_h = D\delta_{g^{-1}h} - \ell(h)\lambda(g)\delta_h \\ &= \ell(g^{-1}h)\delta_{g^{-1}h} - \ell(h)\delta_{g^{-1}h} = \ell(g^{-1}h) - \ell(h)\delta_{g^{-1}h}\end{aligned}$$

So $||[D, \lambda(g)]\delta_h|| \leq |\ell(g^{-1}h) - \ell(h)|$. Actually we have that

$$\sup_g |\ell(g^{-1}h) - \ell(h)| = \ell(g) .$$

In fact, suppose that $\ell(g^{-1}h) \leq \ell(h)$. We have that $|\ell(g^{-1}h) - \ell(h)| = \ell(g^{-1}h) - \ell(h)$. By using the triangle inequality we get that $\ell(g^{-1}h) \leq \ell(g^{-1}) + \ell(h) = \ell(g) + \ell(h)$ which gives $|\ell(g^{-1}h) - \ell(h)| \leq \ell(g)$. Suppose instead that $\ell(g^{-1}h) \geq \ell(h)$. We have

[2] Actually the definition of length function makes sense for all groups, usually one requires that it is continuous.

that $|\ell(g^{-1}h) - \ell(h)| = \ell(h) - \ell(g^{-1}h)$. By using the triangle inequality we get that $\ell(h) = \ell(gg^{-1}h) \le \ell(g) + \ell(g^{-1}h)$ which gives $|\ell(g^{-1}h) - \ell(h)| \le \ell(g)$. The reverse inequality is clear since with $h = 1$ we get $|\ell(g^{-1}h) - \ell(h)| = |\ell(g^{-1}1) - \ell(1)| = \ell(g)$. From this it is easy to see that the commutator is bounded.

The property concerning the compactness of the resolvent is clear since we see that the operator D has an orthonormal bases of eigenvectors (i.e. the canonical basis of $\ell^2(\Gamma)$ with eigenvalues going to zero as g goes to infinity.

We mention that since $C^*(\Gamma) = C(\hat{\Gamma}) = C(\mathbb{T})$ we can see that this construction is actually a generalization of the former example (actually we in the former case do not chose a length function, but a function $\underline{n}$ defined by $\underline{n} \cdot m := m$)

Example 1.7.3. *(The UHF algebra)* The third example is the UHF algebra of type r^∞. Christensen and Ivan [CI06] introduced a family of spectral triples for AF-algebras (inductive limits of finite dimensional C^*-algebras). In the following chapters we will consider study the case of UHF algebras, so we briefly discuss the definition of spectral triple for this algebras.

Consider the projection $P_n : L^2(\mathcal{A}, \tau) \to L^2(M_n, \mathrm{Tr})$, where $\mathrm{Tr} : M_r(\mathbb{C}) \to \mathbb{C}$ is the unique normalized trace, and define

$$\begin{aligned} Q_n &= P_n - P_{n-1}, \quad n \ge 0, \\ E(x) &= \tau(x) 1_{\mathcal{A}} . \end{aligned}$$

For any $s > 1$, Christensen and Ivan ([CI06]) defined the following spectral triple for the algebra $UHF(r^\infty) \overset{def}{=} \mathcal{A}$

$$(\mathcal{L}, L^2(\mathcal{A}, \tau), D = \sum_{n \ge 0} r^{ns} Q_n)$$

where $\mathcal{L}$ is the algebra consisting of the elements of $\mathcal{A}$ with bounded commutator with D. The $*$-algebra $\mathcal{L}$ can be chosen as the algebraic inductive limit, i.e. $\mathrm{alg} \varinjlim M_r(\mathbb{C})^{\otimes n}$. It is enough to show that the elements of the following form have bounded commutator with D

$$x_n = I_{[0,n-1]} \otimes b \otimes I_{[n,+\infty]} \quad n \ge 0$$

It can be easily seen that

$$[Q_k, x_n] = \begin{cases} 0 & \text{if } k > n \\ \mathrm{id}_{0,k-1} \otimes (b\mathrm{Tr}(\cdot) - \mathrm{Tr}(b\cdot)) \otimes \tau & \text{if } k = n \\ \mathrm{id}_{0,k-1} \otimes F \otimes \left(\bigotimes_{i=k+1}^{-n-1} \mathrm{Tr}(\cdot) \right) \otimes (\mathrm{Tr}(b\cdot) - b\mathrm{Tr}(\cdot)) \otimes \tau & \text{if } k < n. \end{cases}$$

We have that

$$||[D, x_n]|| \le 2||b|| \left(\sum_{k=0}^{n} r^{ks} \right) < \infty$$

We mention that it was proved that for any such value of the parameter s, this spectral triple induces a metric which defines a topology equivalent to the weak-* topology on the state space ([CI06, Theorem 3.1]).

Example 1.7.4. (Cantor set) Here we introduce a spectral triple for a fractal: the Cantor set, [GI01]. This fractal can be defined as the product $\prod_{i\geq 1}\{0,2\}$, where space $\{0,2\}$ is considered with the discrete topology. It can also be identified with a subset K of the interval $[0,1]$ by using the map

$$\prod_{i\geq 1}\{0,2\} \ni \{x_n\}_{n\geq 1} \longmapsto \sum_{n\geq 1} \frac{x_n}{3^n}\,.$$

Consider the Hilbert space $H := \ell^2(\{a_n, b_n\})$ where $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are sequences such that $b_n - a_n \leq b_{n-1} - a_{n-1}$ and $\sum_n b_n - a_n = 1$, with $\bigsqcup_n (a_n, b_n) = [0,1] \setminus K$. Let D be the following operator

$$D := \frac{1}{b_n - a_n}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let $\mathcal{L} := \mathrm{Lip}(K)$. The algebra $C(K)$ acts faithfully on $\ell^2(\{a_n, b_n\})$ by left multiplication, namely by $(f\eta)(x) = f(x)\eta(x)$ for all $x \in \{a_n, b_n\}$, $f \in C(K)$, $\eta \in \mathrm{Lip}(K)$. Then it can proved that (A, H, D) is a spectral triple. See e.g. [GI01] for further details.

We end this section with the definition of semifinite spectral triple. We will furnish examples of this kind of spectral triple in the following chapter.

Definition 1.7.5. Let $\mathcal{A}$ be a unital C*-algebra. An odd semifinite spectral triple $(\mathcal{L}, \mathcal{H}, D; \mathcal{A}, \tau)$ on $\mathcal{A}$, with respect to a semifinite von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ endowed with a n.s.f. trace τ, is given by a unital, norm-dense, *-subalgebra $\mathcal{L} \subset \mathcal{A}$, a (separable) Hilbert space $\mathcal{H}$, a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\pi(\mathcal{A}) \subset \mathcal{M}$, and an unbounded self-adjoint operator $D \widehat{\in} \mathcal{M}$ such that

(1) $(1 + D^2)^{-1}$ is a τ-compact operator, *i.e.* $\tau(\chi_{(\lambda,+\infty)}((1 + D^2)^{-1})) \to 0$, $\lambda \to +\infty$,

(2) $\pi(a)(\mathrm{dom}D) \subset \mathrm{dom}D$, and $[D, \pi(a)] \in \mathcal{M}$, for all $a \in \mathcal{L}$.

The triple $(\mathcal{L}, \mathcal{H}, D; \mathcal{M}, \tau)$ is even if, in addition,

(3) there is a self-adjoint unitary operator (*i.e.* a $\mathbb{Z}_2$-grading) $\Gamma \in \mathcal{M}$ such that $\pi(a)\Gamma = \Gamma\pi(a)$, $\forall a \in \mathcal{A}$, and $D\Gamma = -\Gamma D$.

The triple $(\mathcal{L}, \mathcal{H}, D; \mathcal{M}, \tau)$ is finitely summable if, in addition,

(4) there is $d > 0$ such that $\tau((1 + D^2)^{-d/2}) < +\infty$.

1.8 Compact quantum metric spaces

In this section we recall the definition and some fundamental properties of compact quantum metric spaces. We will need these notions in the following chapter. Since we will actually consider the metric structure associated to a spectral triple, we will work in the realm of C^*-algebras. We mention that it is also possible to define a compact quantum metric space in a more general setting by working with order-unit spaces, however since we do not need such generality we restrict to the particular case of interest. The results mentioned in this brief section about Compact Quantum Metric Spaces can be found in [Rie99b, Rie04, Rie98], the interested reader is referred to these papers and to the references therein.

Let A be a unital C^*-algebra and let $L : A \to \mathbb{R}^+$ be a seminorm, i.e. a map such that $L(a + b) \leq L(a) + L(b)$ and $L(\lambda a) = |\lambda| L(a)$ for all $a, b \in A$, $\lambda \in \mathbb{C}$.

The state space $S(A)$ is the vector space of continuous linear functionals $\psi : A \to \mathbb{C}$ such that $\psi(1) = 1$ and $\psi(a) \geq 0$ for all $a \geq 0$. We observe that L induces a pseudo-metric on the state space

$$\rho_L(\psi, \mu) := \sup\{|\psi(a) - \mu(a)| \; : \; L(a) \leq 1\} \quad \forall \psi, \mu \in S(A) \, .$$

Definition 1.8.1. The seminorm L is called a Lip-norm if

1. $L(1) = 0$;
2. $L(a) = L(a^*)$ for all $a \in A$;
3. The topology on the state space $S(A)$ induced by the pseudo-metric ρ_L coincides with the weak-$*$ topology.

We observe that the third condition implies that if $L(a) = 0$, then $a \in \mathbb{C}1_A$. In fact, let $a \in A \setminus \mathbb{C}1_A$ be such that $L(a)$ and consider two states ψ, μ such that $\psi(a) \neq \mu(a)$ Now consider the sequence $a_n := na$ (clearly $L(a_n) = L(na) = nL(a) = 0$). We have that

$$\rho_L(\psi, \mu) \geq |\psi(a_n) - \mu(a_n)| = |\psi(na) - \mu(na)| = n|\psi(a) - \mu(a)| \to +\infty \, .$$

However, the state space $S(A)$ is compact, the metric must have finite values for all pairs of states, so the former computation yields a contradiction.

Moreover, the topology induced by the pseudo-metric ρ_L on $S(A)$ is strictly finer that the weak-$*$ topology, [Rie98, Proposition 1.4].

Admittedly the third condition of Defintion 1.8.1 can be hard to be proven. The following theorem gives an equivalent formulation of this property that we will use in the following chapter.

Theorem 1.8.2. *Set $\tilde{A} := A/\mathbb{C}1$ and let $\pi : A \to \tilde{A}$ the natural projection. Consider a seminorm L on A such that $L(1) = 0$. Set $B_1 = \{a \in A \mid L(a) \leq 1\}$. Then*

1. *ρ_L gives $S(A)$ a finite diameter if and only if $\pi(B_1)$ is bounded for $\|\cdot\|^{\sim}$;*
2. *the topology induced by ρ_L coincides with the weak-$*$ topology if and only if $\pi(B_1)$ is totally bounded for $\|\cdot\|^{\sim}$.*

Here we mention the following useful result.

Proposition 1.8.3. *(*[Rie98]*) Let A be a unital C^*-algebra and let L and M two densely defined seminorms. If $L(a) \leq M(a)$ for all $a \in A$, then $\rho_M(\psi, \mu) \leq \rho_L(\psi, \mu)$ for all ψ, $\mu \in S(A)$. In particular, if ρ_L is finite, then so is ρ_M.*

The following immediate corollary is an easy consequence of the fact that a continuous bijection between compact Hausdorff spaces has continuous inverse.

Corollary 1.8.4. *Let A be a unital C^*-algebra and let L and M two densely defined seminorms. Suppose that $L(a) \leq M(a)$ for all $a \in A$. If ρ_L induces the weak-$*$ topology on $S(A)$, , then so does ρ_M.*

We will mainly be interested in a class of seminorms, namely the ones induced by a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$: $L_{\mathcal{D}}(a) := \|[\mathcal{D}, a]\|$. This kind of seminorm might seem special, nevertheless it can be shown that every compact quantum metric space can be described within this framework. The following discussion shows the universality of this approach, [Rie04, Section 4, p. 9]. Consider a compact metric space (X, ρ). Set $Z := \{(x, y) \in X \times X \mid x \neq y\}$. Consider any positive measure μ on X whose support is all X. Let $\omega := (\mu \times \mu) \restriction_Z$ and $A := C(X)$. We have a faithfull representation of A on $H := L^2(Z, \omega)$ defined as $(f\xi)(x, y) := f(x)\xi(x, y)$ for all $f \in A$, $\xi \in H$. Define $(D\xi)(x, y) := \xi(y, x)/\rho(x, y)$ for those $\xi \in H$ such that $D\xi \in H$. It can be shown that the associated seminorm L and the corresponding metric induce the same topology as ρ (see [Rie04] and the reference therein for further details).

We mention another example of compact quantum metric space, [Rie98, Section 2]. Let G be a compact group and let $\ell : G \to \mathbb{R}$ be a (continuous) length function. Given a strongly continuous action α of a compact group G on a C^*-algebra $A \subset \mathcal{B}(\mathcal{H})$ we may define the seminorm on A

$$L(a) := \sup \left\{ \frac{\|\alpha_x(a) - a\|}{\ell(x)} \ : \ x \neq e \right\}$$

The condition $L(a) = L(a^*)$ follows from the fact that α_x is $*$-homomorphism and that $\|a^*\| = \|a\|$. We recall that an action is called ergodic if the fixed point C^*-algebra $A^G = \mathbb{C}$. It is clear that property (1) of Definition 1.8.1 holds if and only if the action $\alpha : G \to \mathrm{Aut}(A)$ is ergodic.

Now we introduce the Gromov-Hausdorff distance. We start with the Hausdorff distance, [Rie04, p.10]. Let (Z, ρ) be a compact metric space. For a subset Y of Z and $r \in \mathbb{R}^+$, we define the open r-neighbourhood by

$$\mathcal{N}_r^\rho(Y) := \{z \in Z \mid \exists y \in Y \text{ such that } \rho(z, y) < r\} .$$

The Hausdorff distance between two closed subset X and Y of Z is then defined as

$$\mathrm{dist}_H^\rho(X, Y) := \inf\{r \mid Y \subset X \subset \mathcal{N}_r^\rho(Y), Y \subset \mathcal{N}_r^\rho(X)\} .$$

Given a metric space (X, ρ) denote by $\mathcal{M}(X, \rho)$ the set of non-empty compact subset of X. It can actually be shown that

- $\mathrm{dist}_H^\rho(\cdot, \cdot)$ is a metric on the set of non-empty compact subset of X;
- $\mathcal{M}(X, \rho)$ is complete if and only if X is complete;

- $\mathcal{M}(X,\rho)$ is compact if and only if X is compact.

Gromov generalized the notion of Hausdorff distance to general compact spaces X and Y, not necessarily contained in a bigger set Z. Consider two compact metric spaces (X,ρ_X) and (Y,ρ_Y). The disjoint union $X \sqcup Y$ can be endowed with a family of metrics such that the restriction to X and Y is ρ_X and ρ_Y, respectively. Denote this set by $\mathcal{M}(\rho_X,\rho_Y)$. Then the Gromov-Hausdorff distance is defined as

$$\mathrm{dist}^{\rho}_{GH}(X,Y) := \inf\{\mathrm{dist}^{\rho}_{H}(X,Y), \rho \in \mathcal{M}(\rho_X,\rho_Y)\}\ .$$

We observe that by definition for any two compact metric spaces we have that $\mathrm{dist}^{\rho}_{H}(X,Y) \leq \mathrm{dist}^{\rho}_{GH}(X,Y)$. It can be proved that given two compact metric space (X,ρ) and (Y,σ), then $\mathrm{dist}^{\rho}_{GH}(X,Y) = 0$ if and only if X and Y are isometric.

Now we take care of the case of the generalization to the framework of compact quantum metric spaces. Let (A,L_A) and (B,L_B) be compact quantum metric spaces. Denote by $\mathcal{M}(L_A,L_B)$ the set of Lip-norms on $A \oplus B$ whose quotient seminorm to A and B are L_A and L_B, respectively. This means that if $L \in \mathcal{M}(L_A,L_B)$, then

$$L_A(a) = \inf\{L(a,b), b \in B\}$$

and

$$L_B(a) = \inf\{L(a,b), a \in A\}\ .$$

The distance between the compact quantum metric spaces (A,L_A) and (B,L_B) is defined as

$$\mathrm{dist}_q(X,Y) := \inf\{\mathrm{dist}^{\rho_L}_{H}(S(A),S(B)) \mid L \in \mathcal{M}(L_A,L_B)\}$$

where we identified $S(A)$ and $S(B)$ with a subset of $S(A \oplus B)$.

1.9 Functions of positive type

In this short section we recall the definition of functions of positive type and the correspondence with unitary representations.

Definition 1.9.1. Let Γ be a group. A function $\varphi : \Gamma \to \mathbb{C}$ is called of positive type if for all $r \in \mathbb{N}$, all $g_1,\dots,g_r \in \Gamma$ and all $a_1,\dots a_r \in \mathbb{C}$, it holds

$$\sum_{i,j=1}^{r} a_i \overline{a_j} \varphi(g_i g_j^{-1}) \geq 0\ .$$

We describe an example of a function of positive type. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation and let $v \in \mathcal{H}$. Then the function $\varphi(g) := \langle \pi(g)v, v\rangle_{\mathcal{H}}$ is a function of positive type. The following theorem says that this is the only example of function of positive type.

Theorem 1.9.2. *Let $\varphi : \Gamma \to \mathbb{C}$ be a function of positive type. Then there exists a triple $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ consisting of a cyclic unitary representation $\pi_\varphi : \Gamma \to \mathcal{U}(\mathcal{H}_\varphi)$ and a cyclic vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that*

$$\varphi(g) = \langle \pi_\varphi(g)\xi_\varphi, \xi_\varphi \rangle \quad \forall g \in \Gamma \ .$$

Proof. Denote by $\mathbb{C}[\Gamma]$ the space of finite linear combinations of elements of Γ. We define the following sesquilinear form on $\mathbb{C}[\Gamma]$

$$\langle \sum_{i\in\Gamma} a_i g_i, \sum_{i\in\Gamma} b_i g_i \rangle := \sum_{i\in\Gamma} a_i \overline{b_j} \varphi(g_i g_j^{-1}) \ .$$

which is always non-negative by the definition of function of positive type. Let N_φ be the subspace consisting of finite formal linear combinations $\sum_{i\in\Gamma} a_i g_i$ such that $\langle \sum_{i\in\Gamma} a_i g_i, \sum_{i\in\Gamma} a_i g_i \rangle = 0$. Now $\mathbb{C}[\Gamma]/N_\varphi$ is a pre-Hilbert space. Let $\mathcal{H}_\varphi$ be its completion and denote by $\langle \cdot, \cdot \rangle_\varphi$ the induced inner product. The pair $(\mathcal{H}_\varphi, \langle \cdot, \cdot \rangle_\varphi)$ is a Hilbert space. Now we set $\xi_\varphi := [e] \in \mathcal{H}_\varphi$, with e being the neutral element of Γ. We define π_φ as follows

$$\pi_\varphi(x) \left(\sum_{i\in\Gamma} a_i g_i \right) := \sum_{i\in\Gamma} a_i \ (g_i x) \quad x \in \Gamma \ .$$

where $\sum_{i\in\Gamma} a_i g_i \in \mathcal{H}_\varphi$. This is a unitary representation. Indeed,

$$\begin{aligned}
\langle \sum_{i\in\Gamma} a_i g_i x, \sum_{i\in\Gamma} a_i g_i x \rangle_\varphi &= \sum_{i,j\in\Gamma} a_i \overline{a_j} \varphi([g_i x][g_j x]^{-1}) \\
&= \sum_{i,j\in\Gamma} a_i \overline{a_j} \varphi(g_i x x^{-1} g_j^{-1}) \\
&= \sum_{i,j\in\Gamma} a_i \overline{a_j} \varphi(g_i g_j^{-1}) \\
&= \langle \sum_{i\in\Gamma} a_i g_i, \sum_{i\in\Gamma} a_i g_i \rangle_\varphi \ .
\end{aligned}$$

□

Chapter 2

Spectral triples for noncommutative solenoidal spaces from self-coverings

In this chapter we introduce the definition of noncommutative covering with an abelian group of deck transformations. Self-coverings naturally lead us to consider inductive limits of algebras (i.e projective limit of commutative and noncommutative spaces). In fact, given a noncommutative self-covering consisting of a C*-algebra with a unital injective endomorphism $(\mathcal{A}, \alpha)$, we study the possibility of extending a spectral triple on $\mathcal{A}$ to a spectral triple on the inductive limit C^*-algebra, where the inductive family associated with the endomorphism α is

$$\mathcal{A}_0 \xrightarrow{\alpha} \mathcal{A}_1 \xrightarrow{\alpha} \mathcal{A}_2 \xrightarrow{\alpha} \mathcal{A}_3 \ldots, \tag{2.0.1}$$

all the $\mathcal{A}_n$ being $\mathcal{A}$. The algebra $\mathcal{A}_n$ may be considered as the n-th covering of the algebra $\mathcal{A}_0$ w.r.t. the endomorphism α. As a remarkable byproduct, all the spectral triples we construct on the inductive limit C^*-algebra are semifinite spectral triples. We describe in detail some examples, namely the cases of p-dimensional tori, rational rotation algebras (i.e. noncommutative tori associated with a rational parameter), some crossed product C^*-algebras, and the UHF algebra. This chapter is based on the results contained in the paper [AGI].

2.1 Noncommutative coverings w.r.t. a finite abelian group

2.1.1 Spectral decomposition

The aim of this section is to describe a spectral decomposition of an algebra in terms of an action of a finite abelian group. For more details and a general theory the interested reader is referred to [Ped79].

Let $\mathcal{B}$ be a C^*-algebra and Γ be a finite abelian group which acts on $\mathcal{B}$ (we denote the action by γ). Let

$$\mathcal{B}_k := \{b \in \mathcal{B} \text{ s.t. } \gamma_g(b) = \langle k, g\rangle b \quad \forall g \in \Gamma\}, \quad k \in \widehat{\Gamma}.$$

Proposition 2.1.1. *With the above notation,*

1. $\mathcal{B}_h\mathcal{B}_k \subset \mathcal{B}_{hk}$; *in particular each* $\mathcal{B}_k$ *is an* $\mathcal{A}$*-bimodule, where* $\mathcal{A}$ *is the fixed point subalgebra,*
2. *if* $b_k \in \mathcal{B}_k$ *is invertible, then* $b_k^{-1}, b_k^* \in \mathcal{B}_{k^{-1}}$,
3. *each* $b \in \mathcal{B}$ *may be written as* $\sum_{k\in\widehat{\Gamma}} b_k$ *with* $b_k \in \mathcal{B}_k$.

Before proving this proposition we recall that by the Schur orthogonality relations, [Ser12], given Γ a finite abelian group, $\widehat{\Gamma}$ its dual,

$$\sum_{k\in\widehat{\Gamma}} \langle k, g\rangle = \delta_{g,e}\cdot|\Gamma| \qquad \forall g\in\Gamma. \tag{2.1.1}$$

Proof. The first two properties follow by definition. Let us set

$$b_k \equiv E_k(b) \overset{def}{=} \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\langle k^{-1}, g\rangle\gamma_g(b).$$

Then, by (2.1.1),

$$\begin{aligned}\sum_{k\in\widehat{\Gamma}} b_k &= \frac{1}{|\Gamma|}\sum_k\sum_g\langle k^{-1}, g\rangle\gamma_g(b)\\ &= \frac{1}{|\Gamma|}|\Gamma|\sum_g\delta_{g,e}\gamma_g(b) = b.\end{aligned}$$

Finally, b_k belongs to $\mathcal{B}_k$ since, for any $g\in\Gamma$,

$$\begin{aligned}\gamma_g(b_k) &= \frac{1}{|\Gamma|}\sum_{h\in\Gamma}\langle k^{-1}, h\rangle\gamma_g\gamma_h(b)\\ &= \langle k^{-1}, g^{-1}\rangle\frac{1}{|\Gamma|}\sum_{h\in\Gamma}\langle k^{-1}, h\rangle\gamma_h(b)\\ &= \langle k, g\rangle b_k.\end{aligned}$$

□

2.1.2 Noncommutative coverings

Definition 2.1.2. A *finite (noncommutative) covering* with abelian group is an inclusion of (unital) C^*-algebras $\mathcal{A}\subset\mathcal{B}$ together with an action of a finite abelian group Γ on $\mathcal{B}$ such that $\mathcal{A} = \mathcal{B}^\Gamma$. We will say that $\mathcal{B}$ is a covering of $\mathcal{A}$ with deck transformations given by the group Γ.

Let us denote by $M_{\widehat{\Gamma}}(\mathcal{B})$ the algebra of matrices, whose entries belong to $\mathcal{B}$ and are indexed by elements of $\widehat{\Gamma}$. Then, to any $b \in \mathcal{B}$, we can associate the matrix $\widetilde{M}(b) \in M_{\widehat{\Gamma}}(\mathcal{B})$ with the following entries

$$\widetilde{M}(b)_{hk} = b_{h-k}, \quad h, k \in \widehat{\Gamma}.$$

By the definition of b_k the following formula easily follows

$$\widetilde{M}(b)\widetilde{M}(b') = \widetilde{M}(bb'). \tag{2.1.2}$$

The following definition is motivated by Theorem 2.1.9 below.

Definition 2.1.3. We say that the finite covering $\mathcal{A} \subset \mathcal{B}$ w.r.t. Γ is regular if each $\mathcal{B}_k$ has an element which is unitary in $\mathcal{B}$, namely we may choose a map $\sigma : \widehat{\Gamma} \to \mathcal{B}$ such that $\sigma(k) \in \mathcal{U}(\mathcal{B}) \cap \mathcal{B}_k$, with $\sigma(e) = I$.

Remark 2.1.4.

(i) Example 2.1.7 shows this assumption does not always hold.

(ii) In the previous definition, it is enough to ask that each $\mathcal{B}_k$ has an element which is invertible in $\mathcal{B}$. Indeed, if $C \in \mathcal{B}_k$ is invertible, and $C = UH$ is its polar decomposition, then $H = (C^*C)^{1/2} \in \mathcal{A}$. It follows that U is unitary and belong to $\mathcal{B}_k$.

(iii) Regularity also implies that the action is faithful. Indeed, if $g \in \Gamma$ acts trivially, we may find $k \in \widehat{\Gamma}$ such that $\langle k, g\rangle \neq 1$, therefore the equation $\gamma_g(b) = \langle k, g\rangle b$ is satisfied only for $b = 0$, and $\mathcal{B}_k$ does not contain invertible elements.

For regular coverings, we can define an embedding of $\mathcal{B}$ into $M_{\widehat{\Gamma}}(\mathcal{A})$. Set

$$M(b)_{hk} = \sigma(h)^{-1}\widetilde{M}(b)_{hk}\sigma(k) = \sigma(h)^{-1}b_{h-k}\sigma(k), \quad h, k \in \widehat{\Gamma}.$$

It follows from Proposition 2.1.1 that $M(b)_{hk} \in \mathcal{A}$.

Theorem 2.1.5. *Under the regularity hypothesis, the algebra $\mathcal{B}$ is isomorphic to a subalgebra of matrices with coefficients in $\mathcal{A}$, i.e. we have an embedding*

$$\mathcal{B} \hookrightarrow \mathcal{A} \otimes M_{\widehat{\Gamma}}(\mathbb{C}). \tag{2.1.3}$$

Proof. It is easy to show that $M(b^*)_{jk} = (M(b)_{kj})^*$, $\forall b \in \mathcal{B}$, $j, k \in \widehat{\Gamma}$. That the product is preserved, namely

$$M(bb')_{hk} = \sum_j M(b)_{hj}M(b')_{jk}$$

follows easily from (2.1.2). □

In this paper we are mainly interested in self-coverings, namely when there exists an isomorphism $\phi : \mathcal{B} \to \mathcal{A}$ or, equivalently, $\mathcal{A}$ is the image of $\mathcal{B}$ under a unital endomorphism $\alpha = j \circ \phi$, where j is the embedding of $\mathcal{A}$ in $\mathcal{B}$.

Theorem 2.1.6. *Given a (noncommutative) regular self-covering with abelian group Γ, we may construct an inductive family $\mathcal{A}_i$ associated with the endomorphism α as in* [Cun82]. *Then, setting $r = |\widehat{\Gamma}| = |\Gamma|$, we have the following embedding:*

$$\varinjlim \mathcal{A}_i \hookrightarrow \mathcal{A} \otimes UHF(r^{\infty}).$$

Proof. By applying Theorem 2.1.5 j times, we get an embedding of $\mathcal{A}_j$ into $\mathcal{A} \otimes M_r^{\otimes j}$. The result immediately follows. □

The following example shows that the regularity property in Definition 2.1.3 is not always satisfied.

Example 2.1.7. Let $\mathcal{B} = M_3(\mathbb{C})$, $\Gamma = \mathbb{Z}_2 = \{0, 1\}$. We have the following action γ on $\mathcal{B}$: $\gamma_0 = id$, $\gamma_1 = ad(J)$, where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore

$$\mathcal{B}_0 = \mathcal{A} = \mathcal{B}^{\Gamma} = \left\{ x \in \mathcal{B} : x = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} \right\}, \quad \mathcal{B}_1 = \left\{ x \in \mathcal{B} : x = \begin{pmatrix} 0 & a & b \\ c & 0 & 0 \\ d & 0 & 0 \end{pmatrix} \right\}.$$

Hence $\mathcal{B}_1$ has no invertible elements.

2.1.3 Representations

Proposition 2.1.8. *Consider a (noncommutative) regular self-covering $\mathcal{A} \subset \mathcal{B}$ with abelian group Γ.*

(1) *A representation π of $\mathcal{A}$ on a Hilbert space H produces a representation $\tilde{\pi}$ of $\mathcal{B}$ on $H \otimes \mathbb{C}^r$, $r = |\widehat{\Gamma}|$, given by $\tilde{\pi}(b) := [\pi(M(b)_{hk})]_{h,k \in \widehat{\Gamma}} \in M_{\widehat{\Gamma}}(\mathcal{B}(H)) = \mathcal{B}(H \otimes \mathbb{C}^r)$, $\forall b \in \mathcal{B}$.*

(2) *If the representation of $\mathcal{A}$ is induced by a state φ via the GNS mechanism, the corresponding representation of $\mathcal{B}$ on $H \otimes \mathbb{C}^r$ is a GNS representation induced by the state $\tilde{\varphi}$, where $\tilde{\varphi}(b) = \varphi \circ E_{\Gamma}$, and E_{Γ} is the conditional expectation from $\mathcal{B}$ to $\mathcal{A}$. Moreover, the map*

$$\begin{array}{ccc} \mathcal{B} & \to & \mathcal{A} \otimes \mathbb{C}^r \\ b & \mapsto & (a_j)_{j \in \widehat{\Gamma}} \end{array}, \quad a_j = \sigma(j)^{-1} b_j \tag{2.1.4}$$

extends to an isomorphism of the Hilbert spaces $L^2(\mathcal{B}, \tilde{\varphi})$ and $L^2(\mathcal{A}, \varphi) \otimes \mathbb{C}^r$.

Proof. (1) It is a simple computation.

(2) Denoting by ξ_{φ} the GNS vector in H, we set $\tilde{\xi}_{\varphi}$ to be the vector ξ_{φ} in H_e and 0 in the other summands. It is cyclic, because

$$\tilde{\pi}(b)\tilde{\xi}_{\varphi} = \oplus_{k \in \widehat{\Gamma}}\ \sigma(k)^{-1} b_k \xi_{\varphi}.$$

Since ξ_φ is cyclic for $\mathcal{A}$, $\{\sigma(k)^{-1}b_k\xi_\varphi : b_k \in \mathcal{B}_k\}$ is dense in H. It induces the state $\tilde{\varphi}$, since

$$(\tilde{\xi}_\varphi, \tilde{\pi}(b)\tilde{\xi}_\varphi) = (\xi_\varphi, b_e\xi_\varphi) = \varphi\left(\frac{1}{|\Gamma|}\sum_{g\in\Gamma}\gamma_g(b)\right) = \varphi \circ E_\Gamma(b).$$

The isomorphism in (2.1.4) follows by the GNS theorem. □

2.1.4 Finite regular coverings

In this subsection we discuss the relation between our definition of (noncommutative) finite regular covering and the classical notion of regular covering. As a byproduct of an analysis on actions of compact quantum groups, it was proved in [BDCH] that a finite covering is regular iff the "can" map is an isomorphism. More precisely, if X and Y are compact Hausdorff spaces and $\pi : X \to Y$ is a covering map with finite group of deck transformations Γ, X is a regular covering of Y if and only if the canonical map

$$\begin{aligned}\mathrm{can} : C(X) \otimes_{C(Y)} C(X) &\to C(X) \otimes C(\Gamma)\\ f_1 \otimes f_2 &\to (f_1 \otimes 1)\delta(f_2),\end{aligned}$$

is an isomorphism of C*-algebras, where $\delta f = \sum_{g\in\Gamma}\gamma_{g^{-1}}(f) \otimes \chi_g$, $\gamma_g : \Gamma \to Aut(C(X))$ and χ_g denote the action induced by Γ and the characteristic function on elements of Γ, respectively.

The map can for classical coverings makes perfect sense in our case too

$$\mathrm{can} : \mathcal{B} \otimes_\mathcal{A} \mathcal{B} \to \mathcal{B} \otimes C(\Gamma),$$

where $\mathrm{can}(x\otimes y) = (x\otimes 1)\delta(y)$ and $\delta(y) = \sum_{g\in\Gamma}\gamma_{g^{-1}}(y)\otimes\chi_g$. In our framework, however, the canonical map is no longer a morphism of C*-algebras, it is a morphism of $(\mathcal{B}-\mathcal{A})$-bimodules. In fact, this map clearly commutes with the left action of $\mathcal{B}$. Moreover, the right action $\mathcal{A}$ commutes with can since $\delta|_\mathcal{A} = id$. The following theorem shows that, under the regularity property of Definition 2.1.3, the can map is an isomorphism, that is, the regularity property according to [BDCH].

Theorem 2.1.9. *Under the above hypotheses, the map* $\mathrm{can} : \mathcal{B} \otimes_\mathcal{A} \mathcal{B} \to \mathcal{B} \otimes C(\Gamma)$ *is an isomorphism of* $(\mathcal{B}-\mathcal{A})$*-bimodules.*

Proof. The group $\Gamma\times\Gamma$ clearly acts on $\mathcal{B}\otimes_\mathcal{A}\mathcal{B}$, the eigenspaces being $(\mathcal{B}\otimes_\mathcal{A}\mathcal{B})_{j,k} = \{\sigma(j)a\otimes\sigma(k) : a \in \mathcal{A}\}$, $(j,k) \in \widehat{\Gamma}\times\widehat{\Gamma}$. Therefore the elements of $\mathcal{B}\otimes_\mathcal{A}\mathcal{B}$ can be written as

$$z = \sum_{j,k\in\widehat{\Gamma}}\sigma(j)a_{j,k}\otimes\sigma(k) \quad a_{j,k} \in \mathcal{A}.$$

Suppose that $\mathrm{can}(z) = 0$. We want to prove that $z = 0$. Using the fact that $\mathcal{B}$ is the direct sum of its eigenspaces we get

$$\begin{aligned}\mathrm{can}(z) &= \sum_{g\in\Gamma}\sum_{j,k\in\widehat{\Gamma}}\langle g^{-1},k\rangle\sigma(j)a_{j,k}\sigma(k)\otimes\chi_g = 0\\ &\Rightarrow \sum_{j,k\in\widehat{\Gamma}}\langle g^{-1},k\rangle\sigma(j)a_{j,k}\sigma(k) = 0 \quad \forall g \in \Gamma,\end{aligned}$$

where $a_{j,k} \in \mathcal{A}$. Now we show that any $a_{j,k}$ is zero. In fact, multiplying by $\langle g, \ell \rangle$ and summing over $g \in \Gamma$, we get

$$0 = \sum_{g\in\Gamma} \langle g, \ell \rangle \sum_{j,k\in\widehat{\Gamma}} \langle g^{-1}, k \rangle \sigma(j) a_{j,k} \sigma(k) = \sum_{j,k\in\widehat{\Gamma}} \langle g, \ell k^{-1} \rangle \sigma(j) a_{j,k} \sigma(k) = |\Gamma| \sum_{j\in\widehat{\Gamma}} \sigma(j) a_{j,k} \sigma(k),$$

which implies that $a_{j,k} = 0$ for all $j, k \in \widehat{\Gamma}$, so that $z = 0$.

Consider $\sum_{g\in\Gamma} b(g) \otimes \chi_g$, we want to show that it can be obtained as $\mathrm{can}(z)$ for some $z \in \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$. By the above computations, it suffices to solve the following equation, for any $\ell \in \widehat{\Gamma}$,

$$\sum_{g\in\Gamma} \langle g, \ell \rangle b(g) = \sum_{g\in\Gamma} \langle g, \ell \rangle \sum_{j,k\in\widehat{\Gamma}} \langle g^{-1}, k \rangle \sigma(j) a_{j,k} \sigma(k),$$

which, using (2.1.1), may be rewritten as

$$\sum_{g\in\Gamma} \langle g, \ell \rangle b(g) \sigma(k)^{-1} = |\Gamma| \sum_{j\in\widehat{\Gamma}} \sigma(j) a_{j,k}.$$

Since each $b(g)$ is given, the coefficients $a_{j,k}$ can be uniquely determined using again the decomposition of $\mathcal{B}$ in its eigenspaces. □

2.2 Self-coverings of tori

2.2.1 The C*-algebra and its spectral triple

We consider the p-torus $\mathbb{T}^p = \mathbb{R}^p / \mathbb{Z}^p$ endowed with the usual metric, inherited from $\mathbb{R}^p$. On this Riemannian manifold we have the Levi-Civita connection $\nabla^{LC} = d$ and we can define the Dirac operator acting on the Hilbert space $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}^p, dm)$

$$D = -i \sum_{a=1}^{p} \varepsilon^a \otimes \partial^a,$$

where $\varepsilon^a = (\varepsilon^a)^* \in M_{2^{[p/2]}}(\mathbb{C})$ furnish a representation of the Clifford algebra for the p-torus (see [LM16] for more details on Dirac operators). Therefore, we have the following spectral triple

$$(C^1(\mathbb{T}^p), \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}^p, dm), D = -i \sum_{a=1}^{p} \varepsilon^a \otimes \partial^a).$$

2.2.2 The covering

Consider an integer-valued matrix $B \in M_p(\mathbb{Z})$ with $|\det(B)| =: r > 1$. This defines a covering of $\mathbb{T}^p$ as follows. Let us set $\mathbb{T}_1 = \mathbb{R}^p / B\mathbb{Z}^p$ seen as a covering space of $\mathbb{T}_0 := \mathbb{T}^p$. Clearly $\mathbb{Z}^p$ acts on $\mathbb{T}_1$ by translations, the subgroup $B\mathbb{Z}^p$ acting trivially by definition, namely we have an action of $\mathbb{Z}_B := \mathbb{Z}/B\mathbb{Z}^p$ on $\mathbb{T}_1$, which is simply the group of deck trasformations for the covering. We denote this action by γ. We are now in the situation described in the previous section, with $\mathcal{A} = C(\mathbb{T}_0)$ the fixed point algebra of $\mathcal{B} = C(\mathbb{T}_1)$ under the action of $\mathbb{Z}_B$. These algebras can be endowed with the following states, respectively

$$\begin{aligned}\tau_0(f) &= \int_{\mathbb{T}_0} f dm, \quad f \in \mathcal{A},\\ \tau_1(f) &= \frac{1}{|\det(B)|}\int_{\mathbb{T}_1} f dm, \quad f \in \mathcal{B},\end{aligned}$$

where dm is Haar measure.

Proposition 2.2.1. *The GNS representation $\pi_1 : \mathcal{B} \to B(L^2(\mathcal{B}, \tau_1)) = B(L^2(\mathbb{T}_1, dm))$ is unitarily equivalent to the representation $\tilde{\pi}_0$ obtained by $\pi_0 : \mathcal{A} \to B(L^2(\mathcal{A}, \tau_0)) = B(L^2(\mathbb{T}_0, dm))$ according to Proposition 2.1.8.*

Proof. By the GNS theorem it is enough to check that $\tau_1 = \tau_0 \circ E$, where E denotes the conditional expectation from $\mathcal{B}$ to $\mathcal{A}$. This follows from the following observation on the associated measures: they are both probability measures that are traslation invariant, by the results on Haar measures the claim follows. □

In order to apply the results of the previous section, we need to choose unitaries in the eigenspaces $\mathcal{B}_k$, $k \in \widehat{\mathbb{Z}_B}$, namely a map $\sigma : g \in \widehat{\mathbb{Z}_B} \to \mathcal{U}(\mathcal{B}) \cap \mathcal{B}_g$.

With $\mathbb{T}_0 = \mathbb{R}^p/\mathbb{Z}^p$, $\mathbb{T}_1 = \mathbb{R}^p/B\mathbb{Z}^p$, $\mathbb{Z}_B = \mathbb{Z}^p/B\mathbb{Z}^p$ as above, set $A = (B^T)^{-1}$, $\langle x, y\rangle = \exp\left(2\pi i \sum_{a=1}^p x^a y^a\right)$, $x, y \in \mathbb{R}^p$.

Lemma 2.2.2. *With the above notation*

(1) *the cardinality $|\mathbb{Z}_B| = r$,*

(2) *the following duality relations hold: $\widehat{\mathbb{T}}_0 = (\mathbb{R}^p/\mathbb{Z}^p)\widehat{} = \mathbb{Z}^p$, $\widehat{\mathbb{T}}_1 = (\mathbb{R}^p/B\mathbb{Z}^p)\widehat{} = A\mathbb{Z}^p$, $\widehat{\mathbb{Z}_B} = (\mathbb{Z}^p/B\mathbb{Z}^p)\widehat{} = A\mathbb{Z}^p/\mathbb{Z}^p$.*

In particular, the duality $\langle z, g\rangle$, $g \in \mathbb{T}_1, z \in A\mathbb{Z}^p$ induces the duality $\langle k, g\rangle_o$, $g \in \mathbb{Z}_B, k \in \widehat{\mathbb{Z}_B}$, namely if $g \in \mathbb{Z}_B \subset \mathbb{T}_1$, $\langle z, g\rangle = \langle \dot{z}, g\rangle_o$, where $\dot{z}$ denotes the class of z in $\widehat{\mathbb{Z}_B}$. For this reason we drop the subscript o in the following.

Proof. The proofs of the claims are all elementary. We only make some comments on the first one. It is well known that each finite abelian group is the direct sum of cyclic groups and that the order of these groups can be obtained with the following procedure. Let $D = SBT$ the Smith normal form of B, where $S, T \in GL(p, \mathbb{Z})$ and $D = \operatorname{diag}(d_1, \cdots d_p) > 0$. Therefore, we have that $\mathbb{Z}_B = \mathbb{Z}^p/B\mathbb{Z}^p \cong \mathbb{Z}^p/D\mathbb{Z}^p$. As B is invertible, so is D and all the diagonal elements are non-zero. Thus, $\mathbb{Z}_B = \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_p}$ and $|\mathbb{Z}_B| = d_1 \cdot \ldots \cdot d_p = \det(D) = \pm\det(B)$. □

Let us consider the short exact sequence of groups

$$0 \longrightarrow \mathbb{Z}^p \longrightarrow A\mathbb{Z}^p \longrightarrow \widehat{\mathbb{Z}_B} \longrightarrow 0. \tag{2.2.1}$$

Such central extension $A\mathbb{Z}^p$ of $\widehat{\mathbb{Z}_B}$ via $\mathbb{Z}^p$ can be described either with a section $s : \widehat{\mathbb{Z}_B} \to A\mathbb{Z}^p$ or via a $\mathbb{Z}^p$-valued 2-cocycle $\omega(k,k') = s(k) + s(k') - s(k+k')$, see e.g. [Bro12]. We choose the unique section such that, for any $k \in \widehat{\mathbb{Z}_B}$, $s(k) \in [0,1)^p$.

Remark 2.2.3. The mentioned choice of the section s will play a role only later. For the moment, we only note that it implies $s(0) = 0$, hence $\omega(k,0) = 0 = \omega(0,k)$.

The covering we are studying is indeed regular according to Definition 2.1.3, since we may construct the map σ as follows:

$$\sigma(k)(t) := \overline{\langle s(k), t\rangle}, \quad k \in \widehat{\mathbb{Z}_B}, t \in \mathbb{T}_1. \tag{2.2.2}$$

2.2.3 Spectral triples on covering spaces of $\mathbb{T}^p$

Given the integer-valued matrix $B \in M_p(\mathbb{Z})$ as above, if $\mathbb{T}^p$ is identified with $\mathbb{R}^p/\mathbb{Z}^p$, then there is an associated self-covering $\pi : t \in \mathbb{T}^p \mapsto Bt \in \mathbb{T}^p$. We denote by α the induced endomorphism of $C(\mathbb{T}^p)$, i.e. $\alpha(f)(t) = f(Bt)$. Then we consider the inductive limit $\mathcal{A}_\infty = \varinjlim \mathcal{A}_n$ described in (2.0.1), where $\mathcal{A}_n = \mathcal{A}$ for any n.

In the next pages it will be convenient to consider the following isomorphic inductive family: $\mathcal{A}_n$ consists of continuous $B^n\mathbb{Z}^p$-periodic functions on $\mathbb{R}^p$, and the embedding is the inclusion. In this way $\mathcal{A}_\infty$ may be identified with a generalized solenoid C*-algebra (cf. [McC65], [LP13]).

Since $\mathbb{T}_n = \mathbb{R}^p/B^n\mathbb{Z}^p$ is a covering space of $\mathbb{T}_0 := \mathbb{T}^p$, the formula of the Dirac operator on $\mathbb{T}_n$ doesn't change. Therefore, we will consider the following spectral triple

$$(C^1(\mathbb{T}_n), \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_n, \frac{1}{r^n}dm), D = -i\sum_{a=1}^{p} \varepsilon^a \otimes \partial^a).$$

The aim of this section is to describe the spectral triple on $\mathbb{T}_n$ in terms of the spectral triple on $\mathbb{T}_0$. Consider the short exact sequences of groups

$$0 \longrightarrow B^n\mathbb{Z}^p \longrightarrow B^{n-1}\mathbb{Z}^p \longrightarrow \mathbb{Z}_B \longrightarrow 0, \tag{2.2.3}$$
$$0 \longrightarrow A^{n-1}\mathbb{Z}^p \longrightarrow A^n\mathbb{Z}^p \longrightarrow \widehat{\mathbb{Z}_B} \longrightarrow 0, \tag{2.2.4}$$

where $\mathbb{Z}_B$ is now identified with the finite group in (2.2.3), hence is a subgroup of $\mathbb{T}_n$. The central extension $A^n\mathbb{Z}^p$ of $\widehat{\mathbb{Z}_B}$ via $A^{n-1}\mathbb{Z}^p$ can be described either with a section $s_n : \widehat{\mathbb{Z}_B} \to A^n\mathbb{Z}^p$ or via a $A^{n-1}\mathbb{Z}^p$-valued 2-cocycle $\omega_n(k,k') = s_n(k) + s_n(k') - s_n(k+k')$, see e.g. [Bro12]. We choose the unique section such that, for any $k \in \widehat{\mathbb{Z}_B}$, $s_n(k) \in A^{n-1}[0,1)^p$, and observe that this is the same as choosing $s_n(k) = A^{n-1}s_1(k)$. In the same way, the second extension $B^{n-1}\mathbb{Z}^p$ of $\mathbb{Z}_B$ via $B^n\mathbb{Z}^p$ can be described either with a section $\widehat{s}_n : \mathbb{Z}_B \to B^{n-1}\mathbb{Z}^p$ or via a $B^n\mathbb{Z}^p$-valued 2-cocycle $\widehat{\omega}_n(k,k') = \widehat{s}_n(k) + \widehat{s}_n(k') - \widehat{s}_n(k+k')$. We choose the unique section such that, for any $k \in \mathbb{Z}_B$, $\widehat{s}_n(k) \in B^n[0,1)^p$. The following result holds

Proposition 2.2.4. *Any function ξ on $\mathbb{T}_i$ can be decomposed as $\xi = \sum_{k\in\widehat{\mathbb{Z}_B}} \xi_k$, where*

$$\xi_k(t) \equiv E_k(\xi)(t) = \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle -k, g\rangle \xi(t-g), \quad t \in \mathbb{T}_i = \mathbb{R}^p/B^i\mathbb{Z}^p.$$

Moreover, this correspondence gives rise to unitary operators

$$\begin{aligned} v_i : L^2(\mathbb{T}_i, dm/r^i) &\to \sum\nolimits_{k\in\widehat{\mathbb{Z}_B}}^{\oplus} L^2(\mathbb{T}_{i-1}, dm/r^{i-1}) = L^2(\mathbb{T}_{i-1}, dm/r^{i-1}) \otimes \mathbb{C}^r \\ \xi &\mapsto \sum\nolimits_{k\in\widehat{\mathbb{Z}_B}}^{\oplus} \sigma(k)^{-1}\xi_k\,. \end{aligned}$$

The multiplication operator by the element $f \in \mathcal{A}_i$ is mapped to the matrix $M_r(f)$ acting on $L^2(\mathbb{T}_{i-1}, dm/r^{i-1}) \otimes \mathbb{C}^r$ given by

$$M_r(f)_{j,k}(t) = \langle s(j) - s(k), t\rangle f_{j-k}(t), \qquad j,k \in \widehat{\mathbb{Z}_B}.$$

In particular, when f is $B^{i-1}\mathbb{Z}$-periodic, namely it is a function on $\mathbb{T}_{i-1}$, then $M_r(f)_{j,k}(t) = f(t)\delta_{j,k}$, i.e. a function f on $\mathbb{T}_{i-1}$ embeds into $\mathcal{B}(L^2(\mathbb{T}_{i-1}, dm/r^{i-1})) \otimes M_r(\mathbb{C})$ as $f \otimes I$.

Proof. The statement follows from the analysis of Proposition 2.1.8, in particular here $b_k = \xi_k$, $M(b) = M_r(f)$. □

Theorem 2.2.5. *The Dirac operator D_n acting on $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_n, \frac{1}{r^n}dm)$ gives rise to an operator, which we denote by $\widehat{D}_n$, when the Hilbert space is identified with the Hilbert space $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm) \otimes (\mathbb{C}^r)^{\otimes n}$ as above. The Dirac operator $\widehat{D}_n$ has the following form:*

$$\widehat{D}_n = V_n D_n V_n^* = D_0 \otimes I - 2\pi \sum_{a=1}^{p} \varepsilon^a \otimes I \otimes \left(\sum_{h=1}^{n} I^{\otimes h-1} \otimes \operatorname{diag}(s_h(\cdot)^a) \otimes I^{\otimes n-h} \right),$$

where $\operatorname{diag}(s_h(\cdot)^a)_{j,k} = \delta_{j,k} s_j(k)^a$ for $j,k \in \widehat{\mathbb{Z}_B}$, the unitary operator $V_n : \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_n, \frac{1}{r^n}dm) \to \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm) \otimes (\mathbb{C}^r)^{\otimes n}$ is defined as $V_n := I \otimes [(v_1 \otimes \bigotimes_{j=1}^{n-1} I) \circ (v_2 \otimes \bigotimes_{j=1}^{n-2} I) \circ \cdots \circ v_n]$. Moreover, we have the following spectral triple

$$(\mathcal{L}_n := C^1(\mathbb{T}_n), \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm) \otimes (\mathbb{C}^r)^{\otimes n}, \widehat{D}_n).$$

Proof. First of all we prove the formula for $n = 1$. We give a formula for D_1 acting on $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_1, \frac{1}{r}dm) \cong \mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm) \otimes \mathbb{C}^r$. Let us denote by $\{\eta_k\}_{k\in\widehat{\mathbb{Z}_B}}$ a r-tuple of vectors in $\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_0, dm)$, so that $\xi := \sum_{k\in\widehat{\mathbb{Z}_B}} \sigma(k)\eta_k$ is an element in

$\mathbb{C}^{2^{[p/2]}} \otimes L^2(\mathbb{T}_1, \frac{1}{r}dm)$, and $E_k(\xi) = \sigma(k)\eta_k$, $k \in \widehat{\mathbb{Z}_B}$. Then, for any $t \in \mathbb{T}_1$, we get

$$\begin{aligned}
\widehat{D}_1\big(\textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \eta_k(t)\big) &= V_1 D_1 V_1^*\big(\textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \eta_k(t)\big)\\
&= \textstyle\sum^{\oplus}_{j\in\widehat{\mathbb{Z}_B}} \dfrac{1}{r}\langle s(j), t\rangle \displaystyle\sum_{g\in\mathbb{Z}_B} \langle -j, g\rangle D\Big(\sum_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), -t+g\rangle \eta_k(t-g)\Big)\\
&= \textstyle\sum^{\oplus}_{j\in\widehat{\mathbb{Z}_B}} \displaystyle\sum_{k\in\widehat{\mathbb{Z}_B}} \frac{1}{r}\langle s(j), t\rangle \sum_{g\in\mathbb{Z}_B} \langle k-j, g\rangle D\big(\langle s(k), -t\rangle \eta_k(t)\big)\\
&= \textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), t\rangle D\big(\langle s(k), -t\rangle \eta_k(t)\big)\\
&= -i\displaystyle\sum_{a=1}^{p} \textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), t\rangle \varepsilon^a \partial^a\big(\langle s(k), -t\rangle \eta_k(t)\big)\\
&= -i\displaystyle\sum_{a=1}^{p} \textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \varepsilon^a\big(-2\pi i s(k)^a \eta_k(t) + \partial^a \eta_k(t)\big)\\
&= \displaystyle\sum_{a=1}^{p} \Big(-2\pi\varepsilon^a \otimes I \otimes \mathrm{diag}(s(k)^a)_{k\in\widehat{\mathbb{Z}_B}} - i\varepsilon^a \otimes \partial^a \otimes I\Big) \textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \eta_k(t).
\end{aligned}$$

The formula for $n > 1$ can be obtained by iterating the above procedure. □

2.2.4 The inductive limit spectral triple

The aim of this section is to construct a spectral triple for the inductive limit $\varinjlim \mathcal{A}_n$. We begin with some preliminary results. A matrix $B \in M_p(\mathbb{Z})$ is called purely expanding if, for all vectors $v \neq 0$, we have that $\|B^n v\|$ goes to infinity.

Proposition 2.2.6. *Assume* $\det B \neq 0$, $A = (B^T)^{-1}$. *Then the following are equivalent:*

(1) *B is purely expanding,*

(2) $\|A^n\| \to 0$,

(3) *the spectral radius* $\mathrm{spr}(A) < 1$,

(4) $\sum_{n\geq 0} \|A^n\| < \infty$.

Proof. (1) $\Leftrightarrow$ (2) Consider a vector $w = B^n v / \|B^n v\|$, then, from the identity

$$\|B^n w\| = \frac{\|v\|}{\|B^n v\|},$$

we deduce that (1) is equivalent to $\|B^{-n} v\| \to 0$, for all $v \neq 0$. The latter is equivalent to (2) by the identity $(A^n v, u) = (v, B^{-n} u)$, for any vectors u, v.

(2) $\Rightarrow$ (3) We argue by contradiction. Let $\lambda \in \mathrm{sp}(A)$ have modulus $|\lambda| \geq 1$, and consider an associated eigenvector $v \neq 0$. Then, we have that $\|A^n v\| = |\lambda|^n \|v\| \not\to 0$.

(3) $\Rightarrow$ (4) Let $A = C^{-1}(D+N)C$ be the Jordan decomposition of A, where D is the diagonal part, and N the nilpotent one. Then

$$
\begin{aligned}
\|(D+N)^n\| &= \|\sum_{j=0}^{p-1}\binom{n}{j}D^{n-j}N^j\| \le \sum_{j=0}^{p-1}\binom{n}{j}\|D^{n-j}\| \\
&= \sum_{j=0}^{p-1}\binom{n}{j}\mathrm{spr}(A)^{n-j} \le \mathrm{spr}(A)^n\left(\sum_{j=0}^{p-1}n^j\mathrm{spr}(A)^{-j}\right) \qquad (2.2.5)\\
&= \mathrm{spr}(A)^n\frac{(n/\mathrm{spr}(A))^p-1}{n/\mathrm{spr}(A)-1} < \frac{n^p}{n-1}\mathrm{spr}(A)^{n-p},
\end{aligned}
$$

where we used $N^p = 0$, $\|N^j\| \le 1$, $\|D\| = \mathrm{spr}(A)$, so that the series $\sum_{n\ge0}\|A^n\|$ converges.

(4) $\Rightarrow$ (2) is obvious. □

Theorem 2.2.7. *Assume now that B is purely expanding and consider the C^*-algebras $\mathcal{A}_n = C(\mathbb{R}^p/B^n\mathbb{Z}^p)$, which embed into $M_{2^{[p/2]}}(\mathbb{C})\otimes\mathcal{B}(L^2(\mathbb{T}_0,dm))\otimes M_{r^n}(\mathbb{C})$, and the Dirac operators $\widehat{D}_n\widehat{\in}\mathcal{B}(\mathcal{H}_0)\otimes UHF(r^\infty)$, where $\mathcal{H}_0 := \mathbb{C}^{2^{[p/2]}}\otimes L^2(\mathbb{T}_0,dm)$. As a consequence, $\mathcal{A}_\infty$ embeds in the injective limit*

$$\varinjlim \mathcal{B}(\mathcal{H}_0)\otimes M_{r^n}(\mathbb{C}) = \mathcal{B}(\mathcal{H}_0)\otimes \mathrm{UHF}(r^\infty)$$

hence in $\mathcal{B}(\mathcal{H}_0)\otimes\mathcal{R}$, where $\mathcal{R}$ is the injective type II_1 factor. Moreover, the operator $\widehat{D}_\infty$ has the following form:

$$\widehat{D}_\infty = D_0\otimes I - 2\pi\sum_{a=1}^{p}\varepsilon^a\otimes I\otimes\left(\sum_{h=1}^{\infty}I^{\otimes h-1}\otimes\mathrm{diag}(s_h(\cdot)^a)\right).$$

In particular, $\widehat{D}_\infty$ is affiliated to $\mathcal{B}(\mathcal{H}_0)\otimes\mathcal{R} = \mathcal{M}$ and has the form $D_0\otimes I + C$, with $C = C^*\in\mathcal{B}(\mathcal{H}_0)\otimes\mathrm{UHF}(r^\infty)\subset\mathcal{B}(\mathcal{H}_0)\otimes\mathcal{R} = \mathcal{M}$.

Proof. The formula and the fact that $\widehat{D}_\infty$ is affiliated to $\mathcal{M}$ follow from what has already been proved and the following argument. We posed $s_n(k)\in A^{n-1}[0,1)^p$, therefore

$$\max_{k\in\widehat{\mathbb{Z}_B}}\|s_n(k)\| \le \sup_{x\in[0,1)^p}\|A^{n-1}x\| \le \|A^{n-1}\|\sqrt{p}.$$

As a consequence, for any $a\in\{1,\dots,p\}$,

$$\|\,\mathrm{diag}(s_n(k)_a)_{k\in\widehat{\mathbb{Z}_B}}\| = \max_{k\in\widehat{\mathbb{Z}_B}}|s_n(k)_a| \le \max_{k\in\widehat{\mathbb{Z}_B}}\|s_n(k)\| \le \|A^{n-1}\|\sqrt{p}.$$

Recalling that $\widehat{D}_\infty = D_0\otimes I + C$, with $C = 2\pi\sum_{a=1}^{p}\varepsilon^a\otimes I\otimes\left(\sum_{h=1}^{\infty}I^{\otimes h-1}\otimes\mathrm{diag}(s_h(k)^a)\right)$, we get, by Proposition 2.2.6 and the estimate above, that C is bounded and belongs to $M_{2^{[p/2]}}(\mathbb{C})\otimes\mathbb{C}\otimes\mathrm{UHF}(r^\infty)$, while $D_0\widehat{\in}\mathcal{B}(\mathcal{H}_0)$. □

Theorem 2.2.8. *Let $\{(\mathcal{A}_n, \varphi_n) : n \in \mathbb{N} \cup \{0\}\}$ be an inductive system, with $\mathcal{A}_n \cong \mathcal{A}_0$, and $\varphi_n : \mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$ is the inclusion, for all $n \in \mathbb{N}$. Suppose that, for any $n \in \mathbb{N} \cup \{0\}$, there exists a spectral triple $(\mathcal{L}_n, \mathcal{H}_n, \widehat{D}_n)$ on $\mathcal{A}_n$, with $\mathcal{H}_n = \mathcal{H}_0 \otimes (\mathbb{C}^r)^{\otimes n}$, $\widehat{D}_n = D_0 \otimes I + C_n$, $C_n \in \mathcal{B}(\mathcal{H}_0) \otimes M_r(\mathbb{C})^{\otimes n} \subset \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$ is a self-adjoint sequence converging to $C \in \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$, and $\widehat{D}_\infty = D_0 \otimes I + C$. Let d be the abscissa of convergence of ζ_{D_0} and suppose that $res_{s=d}(\tau(D_0^2+1)^{-s/2})$ exists and is finite. Let $\mathcal{L}_\infty := \cup_{n=0}^\infty \mathcal{L}_n$. Then $(\mathcal{L}_\infty, \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}, \mathcal{H}_0 \otimes L^2(\mathcal{R}, \tau), \widehat{D}_\infty)$ is a finitely summable, semifinite, spectral triple, with the same Hausdorff dimension of $(\mathcal{L}_0, \mathcal{H}_0, D_0)$. Moreover, the volume of this noncommutative manifold coincides with the volume of $(\mathcal{L}_0, \mathcal{H}_0, D_0)$, namely the Dixmier trace τ_ω of $(\widehat{D}_\infty^2 + 1)^{-d/2}$ coincides with that of $(D_0^2 + 1)^{-d/2}$ (hence does not depend on ω) and may be written as:*

$$\tau_\omega((\widehat{D}_\infty^2 + 1)^{-d/2}) = \lim_{t\to\infty} \frac{1}{\log t} \int_0^t \left(\mu_{(D_0^2+1)^{-1/2}}(s)\right)^d \, ds.$$

Proof. As for the commutator condition, we observe that for each $f \in \mathcal{L}_n$ we have that $[\widehat{D}_\infty, f]$ is bounded since $[\widehat{D}_n, f]$ is bounded.

We now show that $\widehat{D}_\infty$ has τ-compact resolvent, where τ is the unique f.n.s. trace on $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$. Indeed, on a finite factor, any bounded operator has τ-finite rank, hence is τ-compact. Therefore, since D_0 has compact resolvent in $\mathcal{B}(\mathcal{H}_0)$, $D_0 \otimes I$ has τ-compact resolvent in $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$. We have $(D_0 \otimes I + C + i)^{-1} = [I + (D_0 \otimes I + i)^{-1} C]^{-1} (D_0 \otimes I + i)^{-1} = (D_0 \otimes I + i)^{-1} [I + C(D_0 \otimes I + i)^{-1}]^{-1}$, where $I + C(D_0 \otimes I + i)^{-1}$ and $I + (D_0 \otimes I + i)^{-1} C$ have trivial kernel and cokernel. Indeed $Ran(I + (D_0 \otimes I + i)^{-1} C)^\perp = \ker(I + C(D_0 \otimes I - i)^{-1})$, and $(I + C(D_0 \otimes I \pm i)^{-1})x = 0$ means $(C + D_0 \otimes I)y = \mp iy$ with $y = (D_0 \otimes I \pm i)^{-1}x$, which is impossible since $C + D_0 \otimes I$ is self-adjoint. Moreover, $ker(I + (D_0 \otimes I + i)^{-1} C)$ is trivial. In fact, $(I + (D_0 \otimes I \pm i)^{-1} C)x = 0$ implies that $(D_0 \otimes I + C)x = \mp ix$ which is impossible because $D_0 \otimes I + C$ is self adjoint. Therefore $I + C(D_0 \otimes I + i)^{-1}$ has bounded inverse, hence $D_0 \otimes I + C$ has τ-compact resolvent.

Since D_0 has spectral dimension d, $\mathrm{res}_{s=d}(\tau(D_0^2+1)^{-s/2})$ exists and is finite. Then, applying Proposition 2.7.4, in the appendix, we get $\mathrm{res}_{s=d}(\tau(D_0^2+1)^{-s/2}) = \mathrm{res}_{s=d}(\tau(D_\infty^2 + 1)^{-s/2})$. The result follows by [CRSS07], Thm 4.11. □

Corollary 2.2.9. *Let $(\mathcal{L}_n, \mathcal{H}_n, \widehat{D}_n)$ be the spectral triple on $\mathbb{T}_n$ constructed in Theorem 2.2.5, and let us set $\mathcal{L}_\infty := \cup_{n=0}^\infty \mathcal{L}_n$, $\mathcal{M}_\infty := \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$, $\mathcal{H}_\infty := \mathcal{H}_0 \otimes L^2(\mathcal{R}, \tau)$. Then $(\mathcal{L}_\infty, \mathcal{M}_\infty, \mathcal{H}_\infty, \widehat{D}_\infty)$ is a finitely summable, semifinite, spectral triple, with Hausdorff dimension p. Moreover, the Dixmier trace τ_ω of $(\widehat{D}_\infty^2 + 1)^{-p/2}$ coincides with that of $(D_0^2 + 1)^{-p/2}$ (hence does not depend on ω) and may be written as:*

$$\tau_\omega((\widehat{D}_\infty^2 + 1)^{-p/2}) = \lim_{t\to\infty} \frac{1}{\log t} \int_0^t \left(\mu_{(D_0^2+1)^{-1/2}}(s)\right)^p \, ds.$$

Proof. By construction, $\mathcal{L}_\infty$ is a dense $*$-subalgebra of the C*-algebra $\mathcal{A}_\infty \subset \mathcal{M}_\infty$. The thesis follows from Theorem 2.2.8 and the above results. □

2.3 Self-coverings of rational rotation algebras

2.3.1 Coverings of noncommutative tori

Let A_ϑ be the noncommutative torus generated by U, V with $UV = e^{2\pi i\vartheta}VU$, $\vartheta \in [0,1)$. Given a matrix $B \in M_2(\mathbb{Z})$, $\det B \neq 0$, $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we may consider the C*-subalgebra A_ϑ^B generated by the elements

$$U_1 = U^a V^b, \qquad V_1 = U^c V^d. \tag{2.3.1}$$

We may set $W(n) := U^{n_1}V^{n_2}$ with $n \in \mathbb{Z}^2$. By using the commutation relation between U and V, it is easy to see that

$$\begin{aligned} W(m)W(n) &= e^{-2\pi i\theta m_2 n_1}, \\ W(n)^k &= e^{-\pi i\theta k(k-1)n_1 n_2} W(kn), \qquad \forall k \in \mathbb{Z}. \end{aligned} \tag{2.3.2}$$

Lemma 2.3.1.
(i) $A_\vartheta^B = A_\vartheta \Longleftrightarrow r = |\det B| = 1$.
(ii) $A_\vartheta^B \cong A_{\vartheta'}$, *where* $\vartheta' = r\vartheta$.
(iii) $A_\vartheta^B \cong A_\vartheta$ *iff* $r \equiv_q \pm 1$.

Proof. $(i)(\Leftarrow)$ By using equation (2.3.2) it can be shown that the generators of $(A_\vartheta^B)^{B^{-1}}$ are

$$\begin{aligned} U_2 &= e^{\pi i\vartheta bd(1-a+c)\det B} U, \\ V_2 &= e^{\pi i\vartheta ac(1+b-d)\det B} V. \end{aligned}$$

Hence $A_\vartheta = (A_\vartheta^B)^{B^{-1}} \subset A_\vartheta^B \subset A_\vartheta$, namely these algebras coincide.

(ii) We compute the commutation relations for U_1 and V_1, getting $U_1V_1 = e^{2\pi i \det B\vartheta}V_1U_1$. Since $A_{\det B\vartheta} \cong A_{r\vartheta}$, the statement follows.

(iii) We have $A_\vartheta \cong A_{\vartheta'} \Leftrightarrow \vartheta \pm \vartheta' \in \mathbb{Z} \Leftrightarrow (r \pm 1)\vartheta \in \mathbb{Z}$. This means in particular that $\vartheta = p/q$, for some relatively prime $p, q \in \mathbb{N}$, and $r \equiv_q \pm 1$.

$(i)(\Rightarrow)$ Finally, we observe that $A_\vartheta = A_\vartheta^B \Rightarrow A_\vartheta \cong A_{\vartheta'}$. In the following section (Remark 2.3.2) we show that A_ϑ^B is a proper subalgebra of A_ϑ when $r \neq \pm 1$, thus completing the proof of (i). □

On the one hand, the previous Lemma shows that, setting $\vartheta_n = r^{-n}\vartheta$, the algebras A_{ϑ_n} form an inductive family, where $A_{\vartheta_{k-1}}$ can be identified with the subalgebra $A_{\vartheta_k}^B$ of A_{ϑ_k}. The inductive limit is a noncommutative solenoid according to [LP13, LP16].

On the other hand, since in this paper we are mainly concerned with self-coverings, we will, in the following, consider only the rational case $\vartheta = p/q$, with $r \equiv_q \pm 1$. Possibly replacing B with $-B$, this is the same as assuming $\det B \equiv_q 1$.

2.3.2 The C*- algebra, a spectral triple and the self-covering

A description of A_θ

We are now going to give a description of the rational rotation algebra making small modifications to the description of A_θ, $\theta = p/q \in \mathbb{Q}$, seen in [BEEK92]. Consider the following matrices

$$(U_0)_{hk} = \delta_{h,k} e^{2\pi i(k-1)\theta}, \quad (V_0)_{hk} = \delta_{h+1,k} + \delta_{h,q}\delta_{k,1} \in M_q(\mathbb{C})$$

and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{C}).$$

We have that $U_0 V_0 = e^{2\pi i\theta} V_0 U_0$. Let $n = (n_1, n_2) \in \mathbb{Z}^2$ and set $W_0(n) \overset{def}{=} U_0^{n_1} V_0^{n_2}$, $\tilde{\gamma}_n(f)(t) := \mathrm{Ad}(W_0(Jn))[f(t-n)] = V_0^{n_1} U_0^{-n_2} f(t-n) U_0^{n_2} V_0^{-n_1}$. Since formula (2.3.2) holds whenever two operators satisfy the commutation relation $UV = e^{2\pi i\theta} VU$, the following formula holds

$$W_0(n)^k = e^{-\pi i\theta k(k-1)n_1 n_2} W_0(kn) \qquad \forall k \in \mathbb{Z}. \tag{2.3.3}$$

We have the following description of A_θ (cf. [BEEK92])

$$A_\theta = \{f \in C(\mathbb{R}^2, M_q(\mathbb{C})) \,:\, f = \tilde{\gamma}_n(f), n \in \mathbb{Z}^2\}.$$

This algebra comes with a natural trace

$$\tau(f) := \frac{1}{q} \int_{\mathbb{T}_0} \mathrm{tr}(f(t)) dt,$$

where we are considering the Haar measure on $\mathbb{T}_0$ and $\mathrm{tr}(A) = \sum_i a_{ii}$. We observe that the function $\mathrm{tr}(f(t))$ is $\mathbb{Z}^2$-periodic. The generators of the algebra are

$$U(t_1, t_2) = e^{2\pi i\theta t_1} U_0,$$
$$V(t_1, t_2) = e^{2\pi i\theta t_2} V_0.$$

They satisfy the following commutation relation

$$U(t)^\alpha V(t)^\beta = e^{2\pi i\theta\alpha\beta} V(t)^\beta U(t)^\alpha, \quad \alpha, \beta \in \mathbb{Z}.$$

We set $W(n,t) = U(t)^{n_1} V(t)^{n_2}$, $\forall t \in \mathbb{R}^2$, $n \in \mathbb{Z}^2$, and note that

$$W(m,t)W(n,t) = e^{2i\pi\theta(m,Jn)} W(n,t)W(m,t),$$
$$U(t) = W((1,0),t),$$
$$V(t) = W((0,1),t).$$

We observe that $\tilde{\gamma}_n(f)(t) = \mathrm{Ad}(W(Jn,t))[f(t-n)]$, $\forall t \in \mathbb{R}^2$, $n \in \mathbb{Z}^2$.

A spectral triple for A_θ

Define

$$\mathcal{L}_\theta := \left\{ \sum_{r,s} a_{rs} U^r V^s : (a_{rs}) \in S(\mathbb{Z}^2) \right\},$$

where $S(\mathbb{Z}^2)$ is the set of rapidly decreasing sequences. It is clear that the derivations ∂_1 and ∂_2, defined as follows on the generators, extend to $\mathcal{L}_\theta$

$$\begin{aligned} \partial_1(U^h V^k) &= 2\pi i h U^h V^k \\ \partial_2(U^h V^k) &= 2\pi i k U^h V^k. \end{aligned}$$

Moreover, the above derivations extend to densely defined derivations both on A_θ and $L^2(A_\theta, \tau)$.

We still denote these extensions with the same symbols. We may consider the following spectral triple (see [GBVF00, Section 12.3])

$$(\mathcal{L}_\theta, \mathbb{C}^2 \otimes L^2(A_\theta, \tau), D = -i(\varepsilon^1 \otimes \partial_1 + \varepsilon^2 \otimes \partial_2)),$$

where $\varepsilon^1, \varepsilon^2$ denote the Pauli matrices. In order to fix the notation we recall that the Pauli matrices are self-adjoint, in particular they satisfy the condition $(\varepsilon^k)^2 = I$, $k = 1, 2$.

The noncommutative self-covering

Let $\mathcal{A} \doteq A_\theta$ be a rational rotation algebra, $\vartheta = p/q$, $B \in M_2(\mathbb{Z})$ be a matrix such that $\det B \equiv_q 1$, $r := |\det B| > 1$, and set $C_B = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ the cofactor matrix of B, and $A = (B^T)^{-1}$. Then a self-covering of $\mathcal{A}$ may be constructed in analogy with the construction for the classical torus. Consider the C^*-algebra

$$\mathcal{B} := \{ f \in C(\mathbb{R}^2, M_q(\mathbb{C})) : f = \tilde{\gamma}_{Bn}(f), n \in \mathbb{Z}^2 \}.$$

This algebra is generated by the elements

$$\begin{aligned} U_{\mathcal{B}}(t) &= e^{\pi i \vartheta b d (1-a+c)} e^{2\pi i \theta \langle A e_1, t \rangle} W_0(C_B e_1), \\ V_{\mathcal{B}}(t) &= e^{\pi i \vartheta a c (1+b-d)} e^{2\pi i \theta \langle A e_2, t \rangle} W_0(C_B e_2), \end{aligned} \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.3.4}$$

and can be endowed with a natural trace

$$\tau_1(f) := \frac{1}{q |\det B|} \int_{\mathbb{T}_1} \mathrm{tr}(f(t)) dt, \qquad f \in \mathcal{B}.$$

The action $\tilde{\gamma}$ of $\mathbb{Z}^2$ on $\mathcal{B}$, being trivial when restricted to $B\mathbb{Z}^2$, induces an action of $\mathbb{Z}_B$.

Remark 2.3.2. The algebra $\mathcal{A}$ coincides on the one hand with the fixed point algebra w.r.t. the action of $\mathbb{Z}_B$, and on the other hand with the algebra $\mathcal{B}^B$ constructed as in (2.3.1). In fact, by using (2.3.3), a straightforward computation shows that the elements

U, V that generate $\mathcal{A}$ are given by $U = U_{\mathcal{B}}^a V_{\mathcal{B}}^b$, $V = U_{\mathcal{B}}^c V_{\mathcal{B}}^d$, proving that the inclusion $\mathcal{A} \subset \mathcal{B}$ is a non-abelian self-covering w.r.t. the group $\mathbb{Z}_B$. Since $C(\mathbb{T}_1)$ is the center of $\mathcal{B}$, the action of $\mathbb{Z}_B$ restricts to the action of $\mathbb{Z}_B$ on $C(\mathbb{T}_1)$ described in the previous section. Therefore, the covering we are studying is regular according to Definition 2.1.3, with the same map σ as that for the commutative torus, see (2.2.2). In particular the action of $\mathbb{Z}_B$ is faithful (cf. Remark 2.1.4), hence the inclusion $\mathcal{A} \subset \mathcal{B}$ is strict since $|\mathbb{Z}_B| = |\det B| > 1$.

Proposition 2.3.3. *The GNS representation $\pi_1 : \mathcal{B} \to B(L^2(\mathcal{B}, \tau_1))$ is unitarily equivalent to the representation obtained by $\pi_0 : \mathcal{A} \to B(L^2(\mathcal{A}, \tau))$ according to Proposition 2.1.8.*

Proof. It is enough to prove that $\tau_1 = \tau_0 \circ E$, where E is the conditional expectation from $\mathcal{B}$ to $\mathcal{A}$. We have that

$$\begin{aligned}
\tau_0[E(f)] &= \frac{1}{q}\int_{\mathbb{T}_0} \operatorname{tr}[E(f)(t)] = \\
&= \frac{1}{qr}\int_{\mathbb{T}_0} \sum_{n\in\mathbb{Z}_B} \operatorname{tr}[\gamma_n(f)(t)] = \\
&= \frac{1}{qr}\int_{\mathbb{T}_0} \sum_{n\in\mathbb{Z}_B} \operatorname{tr}[f(t-n)] = \\
&= \frac{1}{qr}\int_{\mathbb{T}_1} \operatorname{tr}[f(t)] = \tau_1(f).
\end{aligned}$$

□

2.3.3 Spectral triples on noncommutative covering spaces of A_θ

Given the integer-valued matrix $B \in M_2(\mathbb{Z})$ as above, there is an associated endomorphism $\alpha : A_\theta \to A_\theta$ defined by $\alpha(f)(t) = f(Bt)$. Then, we consider the inductive limit $\mathcal{A}_\infty = \varinjlim \mathcal{A}_n$ described in (2.0.1), where $\mathcal{A}_n = \mathcal{A}$ for any n.

As in Section 2.2, it will be convenient to consider the following isomorphic inductive family: $\mathcal{A}_n$ consists of continuous $B^k\mathbb{Z}^2$-invariant matrix-valued functions on $\mathbb{R}^2$, i.e

$$\mathcal{A}_k := \{f \in C(\mathbb{R}^2, M_q(\mathbb{C})) \,:\, f = \tilde{\gamma}_{B^k n}(f), n \in \mathbb{Z}^2\},$$

with trace

$$\tau_k(f) = \frac{1}{q|\det B^k|}\int_{\mathbb{T}_k} \operatorname{tr}(f(t))dt,$$

and the embedding is unital inclusion $\alpha_{k+1,k} : \mathcal{A}_k \hookrightarrow \mathcal{A}_{k+1}$. In particular, $\mathcal{A}_0 = \mathcal{A}$, and $\mathcal{A}_1 = \mathcal{B}$. This means that $\mathcal{A}_\infty$ may considered as a generalized solenoid C*-algebra (cf. [McC65], [LP13]).

On the n-th noncommutative covering $\mathcal{A}_n$, the formula of the Dirac operator doesn't change and we can consider the following spectral triple

$$(\mathcal{L}_\theta^{(n)}, \mathbb{C}^2 \otimes L^2(\mathcal{A}_n, \tau), D = -i(\varepsilon^1 \otimes \partial_1 + \varepsilon^2 \otimes \partial_2)).$$

The aim of this section is to describe the spectral triple on $\mathcal{A}_n$ in terms of the spectral triple on $\mathcal{A}_0 = A_\theta$.

We will consider the two central extensions (2.2.3) and (2.2.4) (case $p = 2$) with the associated sections $s_n : \widehat{\mathbb{Z}_B} \to A^n\mathbb{Z}^2$ and $\widehat{s}_n : \mathbb{Z}_B \to B^{n-1}\mathbb{Z}^2$ defined earlier.

The following result holds:

Theorem 2.3.4. *Any b in $\mathcal{A}_i$ can be decomposed as $b = \sum_{k\in\widehat{\mathbb{Z}_B}} b_k$, where*

$$b_k(t) = \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle -k, g\rangle \gamma_g(b(t)) \in (\mathcal{A}_i)_k. \tag{2.3.5}$$

Let u_g be the unitary operator on $L^2(\mathcal{A}_i, \tau_i)$ implementing the automorphism γ_g. Then, any $\xi \in L^2(\mathcal{A}_i, \tau_i)$ can be decomposed as $\xi = \sum_{k\in\widehat{\mathbb{Z}_B}} \xi_k$, where

$$\xi_k(t) = \frac{1}{r}\sum_{g\in\mathbb{Z}_B} \langle -k, g\rangle u_g(\xi(t)). \tag{2.3.6}$$

Moreover, this correspondence gives rise to unitary operators $v_i : L^2(\mathcal{A}_i, \tau_i) \to L^2(\mathcal{A}_{i-1}, \tau_{i-1}) \otimes \mathbb{C}^r$ defined by $v_i(\xi) = \{\sigma(k)^{-1}\xi_k\}_{k\in\widehat{\mathbb{Z}_B}}$. The multiplication operator by an element f on $\mathcal{A}_i$ is mapped to the matrix $M_r(f)$ acting on $L^2(\mathcal{A}_{i-1}, \tau_{i-1}) \otimes \mathbb{C}^r$ given by

$$M_r(f)_{h,k}(t) = \langle s(k) - s(h), -t\rangle f_{h-k}(t), \qquad t \in \mathbb{R}^2, h, k \in \widehat{\mathbb{Z}_B}.$$

Proof. The statements follow as in Proposition 2.2.4. □

Theorem 2.3.5. *Set $\mathcal{H}_0 := \mathbb{C}^2 \otimes L^2(\mathcal{A}_0, \tau_0)$. Then the Dirac operator D_n acting on $\mathbb{C}^2 \otimes L^2(\mathcal{A}_n, \tau_n)$ gives rise to the operator $\widehat{D}_n$ when the Hilbert space is identified with $\mathcal{H}_0 \otimes (\mathbb{C}^r)^{\otimes n}$ as above. Moreover, the Dirac operator $\widehat{D}_n$ has the following form:*

$$\widehat{D}_n := V_n D_n V_n^* = D_0 \otimes I - 2\pi \sum_{a=1}^{2} \varepsilon^a \otimes I \otimes \left(\sum_{j=1}^{n} I^{\otimes j-1} \otimes \operatorname{diag}(s_j(k)^a)_{k\in\widehat{\mathbb{Z}_B}} \otimes I^{\otimes n-j} \right),$$

where $V_n : \mathbb{C}^2 \otimes L^2(\mathcal{A}_n, \tau_n) \to \mathcal{H}_0 \otimes (\mathbb{C}^r)^{\otimes n}$ is defined as $V_n := I \otimes [(v_1 \otimes \bigotimes_{j=1}^{n-1} I) \circ (v_2 \otimes \bigotimes_{j=1}^{n-2} I) \circ \cdots \circ v_n]$.

Proof. We prove the formula for $n = 1$, the case $n > 1$ can be obtained by iterating the procedure. Let us denote by $\{\eta_k\}_{k\in\widehat{\mathbb{Z}_B}}$ an element in $\mathbb{C}^2 \otimes L^2(\mathcal{A}_0, \tau_0)$.

$$\begin{aligned}
V_1 D_1 V_1^*\big(\textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \eta_k(t)\big) &= V_1 D_1\Big(\sum_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), -t\rangle \eta_k(t)\Big) \\
&= \textstyle\sum^{\oplus}_{j\in\widehat{\mathbb{Z}_B}} \langle s(j), t\rangle \frac{1}{r} \displaystyle\sum_{g\in\mathbb{Z}_B} \langle -j, g\rangle u_g\Big(\sum_{k\in\widehat{\mathbb{Z}_B}} D\big(\langle s(k), -t\rangle \eta_k(t)\big)\Big) \\
&\overset{(a)}{=} \textstyle\sum^{\oplus}_{j\in\widehat{\mathbb{Z}_B}} \displaystyle\sum_{k\in\widehat{\mathbb{Z}_B}} \langle s(j), t\rangle \frac{1}{r} \sum_{g\in\mathbb{Z}_B} \langle -j, g\rangle D\big(\langle s(k), -t+g\rangle \eta_k(t)\big) \\
&= \textstyle\sum^{\oplus}_{j\in\widehat{\mathbb{Z}_B}} \displaystyle\sum_{k\in\widehat{\mathbb{Z}_B}} \langle s(j), t\rangle \frac{1}{r} \sum_{g\in\mathbb{Z}_B} \langle k-j, g\rangle D\big(\langle s(k), -t\rangle \eta_k(t)\big) \\
&= \textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), t\rangle D\big(\langle s(k), -t\rangle \eta_k(t)\big) \\
&= -i \textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \langle s(k), t\rangle \displaystyle\sum_{a=1}^{2} \langle s(k), -t\rangle \varepsilon^a\big(-2\pi i s(k)^a \eta_k(t) + \partial^a \eta_k(t)\big) \\
&= \left(-i\sum_{a=1}^{2} \varepsilon^a \otimes \partial^a \otimes I - 2\pi \sum_{a=1}^{2} \varepsilon^a \otimes I \otimes \operatorname{diag}(s(k)^a)_{k\in\widehat{\mathbb{Z}_B}}\right) \textstyle\sum^{\oplus}_{k\in\widehat{\mathbb{Z}_B}} \eta_k(t),
\end{aligned}$$

where in (a) we used the facts that $u_g \circ D = D \circ u_g$, and $u_g \equiv id$ on $\mathbb{C}^2 \otimes L^2(\mathcal{A}_0, \tau_0)$. $\square$

2.3.4 The inductive limit spectral triple

Proposition 2.3.6. *The C^*-algebra $\mathcal{A}_n$ embeds into $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{M}_{r^n}(\mathbb{C})$. As a consequence, $\mathcal{A}_\infty$ embeds into the injective limit*

$$\varinjlim \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{M}_{r^n}(\mathbb{C}) = \mathcal{B}(\mathcal{H}_0) \otimes \mathrm{UHF}(r^\infty)$$

hence in $\mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$, where $\mathcal{R}$ is the injective type II_1 factor.

Theorem 2.3.7. *Assume that B is purely expanding and that $\det(B) \equiv_q 1$. Let us set $\mathcal{L}_\theta = \cup_n \mathcal{L}_\theta^{(n)}$, $\mathcal{M} = \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$, and define*

$$\widehat{D}_\infty := D_0 \otimes I - 2\pi \sum_{a=1}^{2} \varepsilon^a \otimes I \otimes \left(\sum_{j=1}^{\infty} I^{\otimes j-1} \otimes \operatorname{diag}(s_j(k)^a)_{k\in\widehat{\mathbb{Z}_B}}\right).$$

Then $(\mathcal{L}, \mathcal{M}, \mathcal{H}_0 \otimes L^2(\mathcal{R}, \tau), \widehat{D}_\infty)$ is a finitely summable, semifinite, spectral triple, with Hausdorff dimension 2. Moreover, the Dixmier trace τ_ω of $(\widehat{D}_\infty^2 + 1)^{-1}$ coincides with that of $(D_0^2 + 1)^{-1}$ (hence does not depend on the generalized limit ω) and may be written as:

$$\tau_\omega((\widehat{D}_\infty^2 + 1)^{-1}) = \lim_{t\to\infty} \frac{1}{\log t} \int_0^t \big(\mu_{(D_0^2+1)^{-1/2}}(s)\big)^2\, ds.$$

Proof. The formula for $\widehat{D}_\infty$ follows from what has already been proved. We want to prove that $\widehat{D}_\infty$ is of the form $D_0 \otimes I + C$, with $C = -2\pi \sum_{a=1}^{2} \varepsilon^a \otimes I \otimes \left(\sum_{j=1}^{\infty} I^{\otimes j-1} \otimes \operatorname{diag}(s_j(k)^a) \right) \in \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$ and $\widehat{D}_\infty \widehat{\in} \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$.

By construction, $\mathcal{L}_\theta$ is a dense $*$-subalgebra of the C*-algebra $\mathcal{A}_\infty \subset \mathcal{M}$. We now prove that $\widehat{D}_\infty$ is affiliated to $\mathcal{M}$. We posed $s_n(k) \in A^{n-1}[0,1)^2$, therefore

$$\max_{k\in\hat{\mathbb{Z}_B}} \|s_n(k)\| \le \sup_{x\in[0,1)^2} \|A^{n-1}x\| \le \|A^{n-1}\|\sqrt{2}.$$

As a consequence, for $a = 1, 2$, $j \in \mathbb{N}$,

$$\|\operatorname{diag}(s_j(k)^a)\| = \max_{k\in\hat{\mathbb{Z}_B}} |s_j(k)^a| \le \max_{k\in\hat{\mathbb{Z}_B}} \|s_j(k)\| \le \|A^{j-1}\|\sqrt{2}.$$

By Proposition 2.2.6 and the estimate above, we get that C is bounded and belongs to $M_2(\mathbb{C}) \otimes \mathbb{C} \otimes \mathrm{UHF}(r^\infty)$, while $D_0 \otimes I \widehat{\in} \mathcal{B}(\mathcal{H}_0) \otimes \mathbb{C}$.

The thesis follows from Theorem 2.2.8 and what we have seen above. □

2.4 Self-coverings of crossed products

2.4.1 The C*-algebra, its spectral triple and the self-covering

The algebra and the noncommutative covering

Let $B \in M_p(\mathbb{Z})$, with $r = |\det(B)| > 1$, and set $A = (B^T)^{-1}$. Consider a finitely summable spectral triple $(\mathcal{L}_\mathcal{Z}, \mathcal{H}, D)$ on the C*-algebra $\mathcal{Z}$ and assume the following:

- there is an action $\rho : G_1 = A\mathbb{Z}^p \to \mathrm{Aut}(\mathcal{Z})$;
- $\sup_{g\in G_1} \|[D, \rho_g(a)]\| < \infty$, for any $a \in \mathcal{L}_\mathcal{Z}$.

Assuming, for simplicity, that $\mathcal{Z} \subset \mathcal{B}(\mathcal{H})$, recall that the crossed product $\mathcal{A}_{G_1} = \mathcal{Z} \rtimes_\rho G_1$ is the C*-subalgebra of $\mathcal{B}(\mathcal{H} \otimes \ell^2(G_1))$ generated by $\pi_{G_1}(\mathcal{Z})$ and U_h, $h \in G_1$, where

$$(\pi_{G_1}(z)\xi)(g) := \rho_g^{-1}(z)\xi(g),$$
$$(U_h\xi)(g) := \xi(g-h), \qquad z \in \mathcal{Z}, g, h \in G_1, \xi \in \ell^2(G_1; \mathcal{H}) \cong \mathcal{H} \otimes \ell^2(G_1).$$

Set $G_0 = \mathbb{Z}^p \subset G_1$. The embedding $\mathcal{Z} \rtimes_\rho G_0 \subset \mathcal{Z} \rtimes_\rho G_1$ is a finite covering with respect to the action $\gamma : \mathbb{Z}_B \to \mathrm{Aut}(\mathcal{Z} \rtimes_\rho G_1)$ defined as

$$\gamma_j\Big(\sum_{g\in G_1} a_g U_g\Big) = \sum_{g\in G_1} \langle \hat{s}(j), g\rangle a_g U_g, \qquad j \in \mathbb{Z}_B,$$

where $\hat{s} : \mathbb{Z}_B \to \mathbb{Z}^p$ is a section of the short exact sequence

$$0 \to B\mathbb{Z}^p \to \mathbb{Z}^p \to \mathbb{Z}_B \to 0.$$

In fact, the fixed point algebra of this action is $\mathcal{A}_{G_0} := \mathcal{Z} \rtimes_\rho G_0$.

The spectral triples

Define the map $\ell : \mathbb{Z}^p \to M_{2^{[p/2]}}(\mathbb{C})$ as $\ell(m) := \sum_{\mu=1}^{p} m_\mu \varepsilon_{\mu+1}^{(p+1)}$, where $\{\varepsilon_i^{(p+1)}\}_{i=1}^{p+1}$ denote the generators of the Clifford algebra $\mathbb{C}l(\mathbb{R}^{p+1})$, and $m \in \mathbb{Z}^p$.

Theorem 2.4.1. *The following triple is a spectral triple for the crossed product* $\mathcal{A}_{G_0} = \mathcal{Z} \rtimes_\rho G_0$

$$(\mathcal{L}_0 = C_c(\mathbb{Z}^p, \mathcal{Z}), \mathcal{H}_0 = \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(\mathbb{Z}^p), D_0 = D \otimes \varepsilon_1^{(p+1)} \otimes I + I \otimes M_\ell).$$

where $C_c(\mathbb{Z}^p, \mathcal{Z}) := \{\sum_{g\in \mathbb{Z}^p} \pi_{G_1}(z_g)U_g : z_g \in \mathcal{L}_\mathcal{Z}, z_g \neq 0$ *for finitely many* $g \in \mathbb{Z}^p\}$, *and* M_ℓ *is the operator of multiplication by the generalized length function* ℓ *(cf.* [GBVF00, p. 333]*). If the Hausdorff dimension* $d(\mathcal{L}_\mathcal{Z}, \mathcal{H}, D) = d$, *then* $d(\mathcal{L}_0, \mathcal{H}_0, D_0) = d + p$.

Proof. The triple in the statement is indeed an iterated spectral triple in the sense of [HSWZ13], sec. 2.4. Equivalently, $\ell(g)$ is a proper translation bounded matrix-valued function (cf. [HSWZ13, Remark 2.15]). For the sake of completeness we sketch the proof of the statement. For the bounded commutator property it is enough to show that the commutators with $\pi_{G_0}(z), z \in \mathcal{L}_\mathcal{Z}$, and with $U_h, h \in G_0$ are bounded. The norm of the first is bounded by $\sup_{g\in G_0} ||[D, \rho_g(a)]||$, which is finite for any $a \in \mathcal{L}_\mathcal{Z}$, the norm of the second is bounded by $||\ell(h)||$. We then explicitly compute the eigenvalues of D_0^2: they are given by $\lambda^2 + ||g||_2^2$, with λ belong to the sequence of eigenvalues of D and $g \in \mathbb{Z}^p$. The compact resolvent property follows. The formula for the dimension can be obtained as in [HSWZ13, Thm.2.7]. □

In a similar way we define the following spectral triple for the crossed product $\mathcal{A}_{G_1} = \mathcal{Z} \rtimes_\rho G_1$

$$(\mathcal{L}_1 = C_c(G_1, \mathcal{Z}), \mathcal{H}_{G_1} = \mathcal{H} \otimes \mathbb{C}^{2^{[p/2]}} \otimes \ell^2(G_1), D_1 = D \otimes \varepsilon_1^{(p+1)} \otimes I + I \otimes M_{\ell_1}).$$

where $\ell_1 : G_1 = A\mathbb{Z}^p \to M_{2^{[p/2]}}(\mathbb{C})$ is defined as $\ell_1(g) := \sum_{\mu=1}^{p} g_\mu \varepsilon_{\mu+1}^{(p+1)}$, $g \in G_1$.

Remark 2.4.2. In this case the triple is not an iterated spectral triple in general, but $\ell_1(g)$ is still a proper translation bounded matrix-valued function. An explicit proof may be given as above.

Regularity and self-covering property

In order to show that the covering is regular according to Definition 2.1.3, we need to define a map σ which takes values in the spectral subspaces of γ. Consider the section $s : \widehat{\mathbb{Z}_B} \to A\mathbb{Z}^p$ defined for the short exact sequence (2.2.1). Define $\sigma : \widehat{\mathbb{Z}_B} \to \mathcal{U}(\mathcal{Z} \rtimes_\rho A\mathbb{Z}^p)$ as

$$\sigma(k) = U_{s(k)}. \tag{2.4.1}$$

We observe that $U_{s(k)} \in (\mathcal{Z} \rtimes_\rho A\mathbb{Z}^p)_k$, $k \in \widehat{\mathbb{Z}_B}$.

We first consider the crossed-product C*-algebras $\mathcal{A}_{G_0}$ and $\mathcal{A}_{G_1}$ as acting on the Hilbert spaces $\mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}} \otimes \ell^2(G_0)$ and $\mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}} \otimes \ell^2(G_1)$. As remarked in section 2.2.3, a short exact sequence of groups can be described either via a section $s : \widehat{\mathbb{Z}_B} \to G_1$ or a 2-cocycle $\omega : \widehat{\mathbb{Z}_B} \times \widehat{\mathbb{Z}_B} \to G_0$, $\omega(j,k) = s(j) + s(k) - s(j+k)$, where $G_1/G_0 = \widehat{\mathbb{Z}_B}$. Since G_1 is a central extension of $\widehat{\mathbb{Z}_B}$ by G_0, the group G_1 may be identified with $(G_0, \widehat{\mathbb{Z}_B})$, with $g \in G_1$ identified with $(g - s \circ p(g), p(g))$, $p(g)$ denoting the projection of g to $\widehat{\mathbb{Z}_B}$. The multiplication rule is given by $(a,b) \cdot (a',b') = (a + a' - \omega(b,b'), b + b')$, [Bro12]. The above choice of the section s implies that in particular $s(0) = 0$, hence $\omega(0,g) = \omega(g,0) = 0$.

Consider the unitary operator

$$
\begin{aligned}
V : \xi \in \ell^2(G_1; \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}}) &\longrightarrow V\xi \in \ell^2(G_0 \times G_1/G_0; \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}}) \\
(V\xi)(m,j) &:= \xi(m + s(j)), \quad m \in G_0, j \in G_1/G_0\,.
\end{aligned}
\tag{2.4.2}
$$

Proposition 2.4.3. *The representation $\pi_{G_1} : \mathcal{Z} \rtimes_\rho G_1 \to \ell^2(G_1; \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}})$ is unitarily equivalent, through V, to the representation obtained by $\pi_{G_0} : \mathcal{Z} \rtimes_\rho G_0 \to \ell^2(G_0; \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}})$ according to Proposition 2.1.8.*

Proof. Since $\mathcal{A}_{G_1}$ is generated by $\pi_{G_1}(z)$, $z \in \mathcal{Z}$, and U_h, $h \in G_1$, it is enough to prove the statement for the generators. Observe that, for any $z \in \mathcal{Z}$, $m,n \in G_0$, $j,k \in G_1/G_0$, $\eta \in \ell^2(G_0 \times G_1/G_0; \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}})$, we have

$$
\begin{aligned}
(V\pi_{G_1}(z)V^*\eta)(n,k) &= (\pi_{G_1}(z)V^*\eta)(n + s(k)) = (\rho^{-1}_{n+s(k)}(z)V^*\eta)(n + s(k)) \\
&= \rho^{-1}_{n+s(k)}(z)\eta(n,k), \\
(VU_{m+s(j)}V^*\eta)(n,k) &= (U_{m+s(j)}V^*\eta)(n + s(k)) = (V^*\eta)(n - m + s(k) - s(j)) \\
&= \eta(n - m - \omega(j, k-j), k - j)\,.
\end{aligned}
$$

In order to obtain the representation of these operators in $M_{G_1/G_0}(\mathcal{B}(\ell^2(G_0; \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}})))$, choose any $\varphi, \psi \in \ell^2(G_0; \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}})$, and denote by $\{e_j\}_{j \in G_1/G_0}$ the canonical basis of $\ell^2(G_1/G_0)$, so that, for any $j,k \in G_1/G_0$, we get

$$
\begin{aligned}
\langle \varphi, (V\pi_{G_1}(z)V^*)_{jk}\psi \rangle &= \langle \varphi \otimes e_j, V\pi_{G_1}(z)V^*(\psi \otimes e_k) \rangle \\
&= \sum_{i \in G_1/G_0} \sum_{n \in G_0} e_j(i)e_k(i) \langle \varphi(n), \rho^{-1}_{n+s(i)}(z)\xi(n) \rangle \\
&= \delta_{jk} \sum_{n \in G_0} \langle \varphi(n), (\pi_{G_0}(\rho^{-1}_{s(j)}(z))\xi)(n) \rangle,
\end{aligned}
$$

which implies that $(V\pi_{G_1}(z)V^*)_{jk} = \delta_{jk}\pi_{G_0}(\rho^{-1}_{s(j)}(z))$; analogously, for $m \in G_0$, $\ell \in$

G_1/G_0,

$$\begin{aligned}\langle\varphi,(VU_{m+s(\ell)}V^*)_{jk}\psi\rangle &= \langle\varphi\otimes e_j, VU_{m+s(\ell)}V^*(\psi\otimes e_k)\rangle\\ &= \sum_{i\in G_1/G_0}\sum_{n\in G_0} e_j(i)e_k(i-\ell)\langle\varphi(n),\psi(n-m-\omega(\ell,i-\ell))\rangle\\ &= \delta_{k,j-\ell}\sum_{n\in G_0}\langle\varphi(n),\psi(n-m-\omega(\ell,j-\ell))\rangle,\end{aligned}$$

which implies that $(VU_{m+s(\ell)}V^*)_{jk} = \delta_{k,j-\ell}U_{m+\omega(\ell,k)}$. On the other hand,

$$M(\pi_{G_1}(z))_{jk} = U^*_{s(j)}E_{j-k}(\pi_{G_1}(z))U_{s(k)} = \delta_{jk}U^*_{s(j)}\pi_{G_1}(z)U_{s(k)},$$

so that

$$\begin{aligned}\langle\varphi, M(\pi_{G_1}(z))_{jk}\psi\rangle &= \delta_{jk}\langle\varphi\otimes e_j, U^*_{s(j)}\pi_{G_1}(z)U_{s(k)}(\psi\otimes e_k)\rangle\\ &= \delta_{jk}\langle\varphi\otimes e_j,\pi_{G_1}(\rho_{-s(j)}(z))(\psi\otimes e_k)\rangle\\ &= \delta_{jk}\sum_{i\in G_1/G_0}\sum_{n\in G_0} e_j(i)e_k(i)\langle\varphi(n),\rho_n^{-1}(\rho^{-1}_{s(j)}(z))\psi(n)\rangle\\ &= \delta_{jk}\sum_{n\in G_0}\langle\varphi(n),(\pi_{G_0}(\rho^{-1}_{s(j)}(z))\psi)(n)\rangle,\end{aligned}$$

which implies that $M(\pi_{G_1}(z))_{jk} = \delta_{jk}\pi_{G_0}(\rho^{-1}_{s(j)}(z))$. Finally,

$$\begin{aligned}M(U_{m+s(\ell)})_{jk} &= U^*_{s(j)}E_{j-k}(U_{m+s(\ell)})U_{s(k)} = \frac{1}{r}\sum_{g\in\mathbb{Z}_B}\langle k-j,g\rangle U^*_{s(j)}\gamma_g(U_{m+s(\ell)})U_{s(k)}\\ &= \frac{1}{r}\sum_{g\in\mathbb{Z}_B}\langle k-j,g\rangle\langle\widehat{s}(g), m+s(\ell)\rangle U^*_{s(j)}U_{m+s(\ell)}U_{s(k)}\\ &= \frac{1}{r}\sum_{g\in\mathbb{Z}_B}\langle k-j+\ell,g\rangle U_{m+s(\ell)+s(k)-s(j)} = \delta_{k,j-\ell}U_{m+\omega(\ell,j-\ell)},\end{aligned}$$

which ends the proof. □

Corollary 2.4.4. *The following diagram commutes:*

$$\begin{array}{ccc}\mathcal{A}_{G_0} & \longrightarrow & \mathcal{A}_{G_1}\\ \downarrow & \circlearrowleft & \downarrow\\ \mathcal{B}(\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_0)) & \longrightarrow & \mathcal{B}(\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_0))\otimes M_r(\mathbb{C})\end{array}\tag{2.4.3}$$

where vertical arrows are the representations, the elements of $\mathcal{A}_{G_1}$ being identified with matrices as in the previous Proposition, and the horizontal arrows are given by the monomorphisms $a\to M_a$, $(M_a)_{j,k} = \delta_{j,k}U^*_{s(j)}aU_{s(j)}$, both for $a\in\mathcal{A}_{G_0}$ and for $a\in\mathcal{B}(\mathcal{H}\otimes\ell^2(G_0))$.

So far we have defined a finite noncommutative covering. In order to obtain a self-covering, $\mathcal{B}$ has to be isomorphic to $\mathcal{A}$, and we have to make further assumptions. Suppose that there exists an automorphism $\beta \in \mathrm{Aut}(\mathcal{Z})$ such that

$$\beta \circ \rho_{Ag} \circ \beta^{-1} = \rho_g, \qquad g \in \mathbb{Z}^p; \tag{2.4.4}$$

The following result tells us that the above algebras yield a noncommutative self-covering.

Proposition 2.4.5. *([Wil07]) Under the above hypotheses, the sub-algebra $\mathcal{A}_{G_0} = \mathcal{Z} \rtimes G_0 \subset \mathcal{A}_{G_1}$ is isomorphic to $\mathcal{A}_{G_1}$, the isomorphism being given by*

$$\alpha : \sum_{g\in\mathbb{Z}^p} a_g U_{Ag} \in \mathcal{A}_{G_1} \mapsto \sum_{g\in\mathbb{Z}^p} \beta(a_g) U_g \in \mathcal{A}_{G_0}.$$

The map α may also be seen as an endomorphism of $\mathcal{A}_{G_1}$.

2.4.2 Spectral triples on covering spaces of $\mathcal{Z} \rtimes_\rho \mathbb{Z}^p$

As above, given an integer-valued matrix $B \in M_p(\mathbb{Z})$ we may define an endomorphism $\alpha : \mathcal{A}_{G_1} \to \mathcal{A}_{G_1}$. Then, we may describe the inductive limit $\mathcal{A}_\infty = \varinjlim \mathcal{A}_n$ where $\mathcal{A}_n = \mathcal{A}_{G_n}$, $G_n = A^n \mathbb{Z}^p$, and the embedding is the inclusion. Endow G_n with the length function $\ell_n : G_n \to M_{2^{\lceil p/2 \rceil}}(\mathbb{C})$ defined as $\ell_n(g) := \sum_{\mu=1}^p g_\mu \varepsilon_{\mu+1}^{(p+1)}$, $g = (g_1, \dots, g_p) \in G_n$ (ℓ_n is indeed a proper translation bounded matrix-valued function, [HSWZ13, Remark 2.15]). Let us observe that $G_n \subset G_{n+1}$ and that $|G_n/G_{n-1}| = |\det B| =: r$.

Let us define the action $\rho^{(n)}$ of G_n on $\mathcal{Z}$ as follows:

$$\rho^{(n)}_{A^n g} = \beta^{-n} \circ \rho_g \circ \beta^n, \quad g \in G_0.$$

Lemma 2.4.6. *For any $m < n$, $g \in G_m$, we have that $\rho^{(n)}_g = \rho^{(m)}_g$, namely the family $\{\rho^{(n)}\}_{n\in\mathbb{N}}$ defines an action ρ of $\cup_n G_n$.*

Proof. From equation (2.4.4), we have

$$\rho^{(m+1)}_g = \beta^{-(m+1)} \circ \rho_{A^{-m-1}g} \circ \beta^{m+1} = \beta^{-m} \circ \rho_{A^{-m}g} \circ \beta^m = \rho^{(m)}_g, \quad g \in G_m.$$

The thesis follows. □

Suppose that

$$\sup_{g\in G_n} \|[D, \rho^{(n)}_g(a)]\| < \infty,$$

for any $a \in \mathcal{L}_n := C_c(G_n, \mathcal{Z})$. Then, the algebra $\mathcal{A}_{G_n}$ has a natural spectral triple

$$(\mathcal{L}_n, \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}} \otimes \ell^2(G_n), D_n = D \otimes \varepsilon_1^{(p+1)} \otimes I + I \otimes M_{\ell_n}).$$

Remark 2.4.7. In order to define a spectral triple for $\mathcal{A}_{G_n}$, we stress that one could make the stronger assumption

$$\begin{aligned} &\|[D,\rho_g(a)]\| \leq c(\rho)\|[D,a]\|, \quad \forall g\in G_1,\\ &\|[D,\beta^k(a)]\| \leq c(\beta)\|[D,a]\|, \quad \forall g\in G_1, k\in\mathbb{Z}, \end{aligned}$$

for any $a\in\mathcal{L}_n$, and some constants $c(\rho)$, $c(\beta)>0$. An even stronger assumption could be

$$\|[D,\rho_g^{(n)}(a)]\| = \|[D,a]\|,$$

for any $a\in\mathcal{L}_n$ and $g\in G_n$.

The aim of this section is to describe the spectral triple on $\mathcal{A}_{G_n}$ in terms of the spectral triple on $\mathcal{A}_{G_0}$. Before proceeding, we observe that as in (2.4.2) we may define a family of unitary operators $v_i:\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_i)\to\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_{i-1})\otimes\ell^2(G_i/G_{i-1})$.

Theorem 2.4.8. *Set $\mathcal{H}_0 := \mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_0)$. Then the Dirac operator D_n acting on $\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_n)$ gives rise to the operator $\widehat{D}_n$ when the Hilbert space is identified with $\mathcal{H}_0\otimes\bigotimes_{i=1}^n\ell^2(G_i/G_{i-1})$ as above, where $G_i/G_{i-1}\cong\widehat{\mathbb{Z}_B}$. The Dirac operator $\widehat{D}_n$ has the following form:*

$$\widehat{D}_n := V_nD_nV_n^* = D_0\otimes I^{\otimes n}+C_n,$$

with $C_n\in\mathcal{B}(\mathcal{H}_0)\otimes M_r(\mathbb{C})^{\otimes n}$ defined, for $\eta\in\ell^2(G_0\times G_1/G_0\times\ldots\times G_n/G_{n-1};\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}})$, as

$$(C_n\eta)(m,j_1,\ldots,j_n) := \sum_{h=1}^n(I\otimes\ell_h(s_h(j_h)))(\eta(m,j_1,\ldots,j_n)),$$

and $V_n:\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_n)\to\mathcal{H}_0\otimes\bigotimes_{j=1}^n\ell^2(G_j/G_{j-1})$ given by $V_n := (v_1\otimes\bigotimes_{j=1}^{n-1}I)\circ(v_2\otimes\bigotimes_{j=1}^{n-2}I)\circ\cdots\circ v_n$.

Proof. For simplicity, we prove the case $n=1$, the case $n>1$ can be proved by iterating the procedure. For any $\eta\in\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_0)\otimes\ell^2(G_1/G_0)\cong\ell^2(G_0\times(G_1/G_0);\mathcal{H}\otimes\mathbb{C}^{2^{\lceil p/2\rceil}})$, we get, for $m\in G_0$, $j\in G_1/G_0$,

$$\begin{aligned} (V_1D_1V_1^*\eta)(m,j) &= (D_1V_1^*\eta)(m+s(j))\\ &= (D\otimes\varepsilon_1^{(p+1)})(V_1^*\eta)(m+s(j)) + (I\otimes\ell_1(m+s(j)))(V_1^*\eta)(m+s(j))\\ &= (D\otimes\varepsilon_1^{(p+1)})(\eta(m,j)) + (I\otimes\ell_1(m+s(j)))(\eta(m,j))\\ &= (D\otimes\varepsilon_1^{(p+1)}+I\otimes\ell_1(m))(\eta(m,j)) + (I\otimes\ell_1(s(j)))(\eta(m,j))\\ &= (D_0\eta)(m,j)+(C_1\eta)(m,j), \end{aligned}$$

where $(C_1\eta)(m,j) := (I\otimes\ell_1(s(j)))(\eta(m,j))$ belongs to $I\otimes\mathcal{B}(\mathbb{C}^{2^{\lceil p/2\rceil}}\otimes\ell^2(G_0\times G_1/G_0))$. We stress that $(C_1\eta)(m,j)$ dos not depend on m because ℓ_1 is a linear map. □

For any $n \in \mathbb{N}_0$ and $x \in \mathcal{L}_n$, set $L_{D_n}(x) := \|[D_n, x]\|$. An immediate consequence of the previous result is that, under a suitable assumption, these seminorms are compatible.

Corollary 2.4.9. *Suppose that*

$$\|[D_0, \mathrm{Ad}(U_g)(x)]\| = \|[D_0, x]\| \quad \forall x \in \cup_n \mathcal{L}_n \ , \ \forall g \in \cup_n G_n \ .$$

Then for any positive integer m, we have that

$$L_{D_{m+1}}(x) = L_{D_m}(x) \quad \forall x \in \mathcal{L}_m \ .$$

Proof. We give the proof for $m = 0$. As in section 2.1.2, the elements in $\mathcal{A}_1$ may be seen as matrices with entries in $\mathcal{A}_0$ acting on $\ell^2(G_1/G_0; \mathcal{H}_0)$. $\mathcal{A}_0$ itself is then embedded in $\mathcal{A}_1$ as diagonal matrices, the matrix $M(x)$ associated with $x \in \mathcal{A}_0$ being $M(x)_{kk} = (\sigma(k)^* x \sigma(k))_{kk} = (U_{-s(k)} x U_{s(k)})_{kk} = (\rho_{-s(k)}(x))_{kk}$, where the action ρ has been naturally extended to $\mathcal{A}_0$. $D_0 \otimes I$ may as well be identified with the diagonal matrix $(D_0 \otimes I)_{kk} = D_0$, therefore their commutator is the diagonal matrix $([D_0, \rho_{-s(k)}(x)])_{kk}$. As for the commutator with the second term of $\widehat{D}_1$, let us describe the Hilbert space as $\ell^2(G_1/G_0; (\mathcal{H} \otimes \ell^2(G_0)) \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}})$. Then both x and C_1 act as diagonal matrices, whose entries jj are $\rho_{-s(j)}(x) \otimes I$ for the first operator and $I \otimes \ell_1(s(j))$ for the second, showing that the corresponding commutator vanishes. The thesis now follows by the assumption. □

2.4.3 The inductive limit spectral triple

The aim of this section is to describe the Dirac operator on $\mathcal{A}_\infty$.

Theorem 2.4.10. *Assume B is purely expanding, set $\mathcal{H}_0 := \mathcal{H} \otimes \mathbb{C}^{2^{\lceil p/2 \rceil}} \otimes \ell^2(G_0)$, $\mathcal{L} = \cup_n \mathcal{L}_n$, $\mathcal{M} = \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$, and define the Dirac operator $\widehat{D}_\infty$ as follows:*

$$\widehat{D}_\infty := D_0 \otimes I_{UHF} + C,$$

where $C = \lim C_n$, $C_n = C_n^* \in \mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$. Then $(\mathcal{L}, \mathcal{M}, \mathcal{H}_0 \otimes L^2(\mathcal{R}, \tau), \widehat{D}_\infty)$ is a finitely summable, semifinite, spectral triple, with the same Hausdorff dimension of $(\mathcal{L}_0, \mathcal{H}_0, D_0)$ (which we denote by d). Moreover, the Dixmier trace τ_ω of $(\widehat{D}_\infty^2 + 1)^{-d/2}$ coincides with that of $(D_0^2 + 1)^{-d/2}$ (hence does not depend on the generalized limit ω) and may be written as:

$$\tau_\omega((\widehat{D}_\infty^2 + 1)^{-d/2}) = \lim_{t \to \infty} \frac{1}{\log t} \int_0^t \left(\mu_{(D_0^2+1)^{-1/2}}(s)\right)^d ds.$$

Proof. The Dirac operator $\widehat{D}_\infty$ is of the form $D_0 \otimes I + C$. First of all, we prove that $\widehat{D}_\infty \widehat{\in} \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$ by showing that $C \in \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{R}$. This claim and the formula follow from what has already been proved and the following argument. Since we posed $s_n(k) \in$

$A^{n-1}[0,1)^p$, by using the properties of the Clifford algebra and the linearity of ℓ_n, we get

$$\begin{aligned}\|\ell_n(\sum_{h=1}^n s_h(j_h))\| &= \|\sum_{h=1}^n \ell_n(s_h(j_h))\| = \|\sum_{h=1}^n s_h(j_h)\| \\ &\leq \sum_{h=1}^n \|s_h(j_h))\| \leq \sqrt{p}\sum_{h=1}^n \|A^{h-1}\|,\end{aligned}$$

so that

$$\|C_n\| = \|\ell_n(\sum_{h=1}^n s_h(j_h))\| \leq \sqrt{p}\sum_{h=1}^n \|A^{h-1}\|.$$

As $\widehat{D}_\infty = D_0 \otimes I + C$, we get, by Proposition 2.2.6 and the estimate above, that C is bounded and belongs to $\mathcal{B}(\mathcal{H}_0) \otimes UHF(r^\infty)$, while $D_0 \widehat{\in} \mathcal{B}(\mathcal{H}_0)$.

Moreover, by construction, $\mathcal{L}$ is a dense $*$-subalgebra of the C*-algebra $\mathcal{A}_\infty \subset \mathcal{M}$. The thesis follows from Theorem 2.2.8 and the above results. □

Remark 2.4.11. The inclusion $G_n \to G_{n+1}$ gives rise to inclusions $i_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$ and $j_n : \mathcal{A}_{G_n} \to \mathcal{A}_{G_{n+1}}$ in such a way that

$$\begin{cases} i_n(a\xi) = j_n(a) i_n(\xi), \\ i_n(D_n\xi) = D_{n+1} i_n(\xi), \end{cases} \qquad a \in \mathcal{A}_{G_n}, \xi \in \mathcal{H}_n.$$

2.5 Self-coverings of UHF-algebras

2.5.1 The C*-algebra, the spectral triple and an endomorphism

We want to consider the C^*-algebra $UHF(r^\infty)$. This algebra is defined as the inductive limit of the following sequence of finite dimensional matrix algebras:

$$\begin{aligned} M_0 &= M_r(\mathbb{C}) \\ M_n &= M_{n-1} \otimes M_r(\mathbb{C}) \quad n \geq 1, \end{aligned}$$

with maps $\phi_{ij} : M_j \to M_i$ given by $\phi_{ij}(a_i) = a_i \otimes 1$. We denote by $\mathcal{A}$ the $UHF(r^\infty)$ C^*-algebra and set $M_{-1} = \mathbb{C}1_{\mathcal{A}}$ in the inductive limit defining the above algebra. The C^*-algebra $\mathcal{A}$ has a unique normalized trace that we denote by τ.

Now we follow [CI06]. Consider the projection $P_n : L^2(\mathcal{A},\tau) \to L^2(M_n, \mathrm{Tr})$, where $\mathrm{Tr} : M_r(\mathbb{C}) \to \mathbb{C}$ is the normalized trace, and define

$$\begin{aligned} Q_n &= P_n - P_{n-1}, \quad n \geq 0, \\ E(x) &= \tau(x) 1_{\mathcal{A}}. \end{aligned}$$

Lemma 2.5.1. *The projection* $Q_n : L^2(\mathcal{A},\tau) \to L^2(M_n,\tau) \ominus L^2(M_{n-1}, \mathrm{Tr})$ *($n \geq 0$) is given by*

$$Q_n(x_0 \otimes \cdots \otimes x_n \otimes \cdots) = x_0 \otimes \cdots \otimes x_{n-1} \otimes [x_n - \mathrm{Tr}(x_n) 1_{M_d(\mathbb{C})}] \tau(x_{n+1} \otimes \cdots),$$

where $\mathrm{Tr} : M_r(\mathbb{C}) \to \mathbb{C}$ *is the normalized trace.*

Proof. The proof follows from direct computations. □

For any $s > 1$, Christensen and Ivan ([CI06]) defined the following spectral triple for the algebra $UHF(r^\infty) \stackrel{def}{=} \mathcal{A}$

$$(\mathcal{L}, L^2(\mathcal{A}, \tau), D_0 = \sum_{n\geq 0} r^{ns} Q_n)$$

where $\mathcal{L}$ is the algebra consisting of the elements of $\mathcal{A}$ with bounded commutator with D_0. It was proved that for any such value of the parameter s, this spectral triple induces a metric which defines a topology equivalent to the weak*-topology on the state space ([CI06, Theorem 3.1]).

Introduce the endomorphism of $\mathcal{A}$ given by the right shift, $\alpha(x) = 1 \otimes x$. Then, according to [Cun82], we may consider the inductive limit $\mathcal{A}_\infty = \varinjlim \mathcal{A}_n$ with $\mathcal{A}_n = \mathcal{A}$ as described in (2.0.1). As in the previous sections, we have the following isomorphic inductive family: $\mathcal{A}_i$ is defined as

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{A}, \\ \mathcal{A}_n &= M_r(\mathbb{C})^{\otimes n} \otimes \mathcal{A}_0, \\ \mathcal{A}_\infty &= \varinjlim \mathcal{A}_i \end{aligned}$$

and the embedding is the inclusion.

We want to stress that this case cannot be described within the framework considered in section 2.1. In fact, it would be necessary to exhibit a finite abelian group that acts trivially on $1_{M_r(\mathbb{C})} \otimes \bigotimes_{i=1}^\infty M_r(\mathbb{C})$ and that has no fixed elements in $M_r(\mathbb{C}) \otimes 1_{\bigotimes_{i=1}^\infty M_r(\mathbb{C})}$. However, since all the automorphisms of $M_r(\mathbb{C})$ are inner, there cannot be any such group.

2.5.2 Spectral triples on covering spaces of UHF-algebras

Each algebra $\mathcal{A}_p$ has a natural Dirac operator (the one considered earlier)

$$(\mathcal{L}^p, H = L^2(\mathcal{A}_p, \tau), D_p = \sum_{n\geq -p} r^{ns} Q_n),$$

where $\mathcal{L}_p$ is the algebra formed of the elements of $\mathcal{A}_p$ with bounded commutator.

The spectral triple on $\mathcal{A}_1$

We are going to describe the Dirac operator on the first covering.

Lemma 2.5.2. *Let $\xi_1 \otimes \xi_\infty \in L^2(\mathcal{A}, \tau)$, $\xi_1 \in M_r(\mathbb{C})$, we have that*

$$Q_n(\xi_1 \otimes \xi_\infty) = \begin{cases} (1 \otimes Q_{n-1})(\xi_1 \otimes \xi_\infty) & \text{if } n > 0 \\ (F \otimes Q_{n-1})(\xi_1 \otimes \xi_\infty) & \text{if } n = 0, \end{cases}$$

where $F : M_r(\mathbb{C}) \to M_r(\mathbb{C})^\circ$ is defined by $F(x) = x - tr(x)$, and $M_r(\mathbb{C})^\circ$ are the matrices with trace 0.

Proposition 2.5.3. *The following relation holds:*

$$D_1 = r^{-s} F \otimes E + I \otimes D_0.$$

Proof. Let $e_{ij} \otimes x \in \mathcal{D}(D_1) \subset L^2(\mathcal{A}_1, \tau)$. We have that

$$\begin{aligned} D_1(e_{ij} \otimes x) &= \sum_{n \geq -1} r^{ns} Q_n(e_{ij} \otimes x) = \\ &= r^{-s} F e_{ij} \otimes Ex + \sum_{n \geq 0} r^{ns} e_{ij} \otimes (Q_n x) = \\ &= [r^{-s} F \otimes E + I \otimes D_0](e_{ij} \otimes x). \end{aligned}$$

The thesis follows by linearity. □

The spectral triple on $\mathcal{A}_n$ and the inductive limit spectral triple

In this section we will consider the Dirac operators on $\mathcal{A}_n$ and $\mathcal{A}_\infty$.

Theorem 2.5.4. *The Dirac operator D_n has the following form*

$$D_n = I^{\otimes n} \otimes D_0 + \sum_{k=1}^{n} r^{-sk} I_r^{n-k} \otimes F \otimes E. \tag{2.5.1}$$

Proof. Let $x \in \mathcal{D}(D_n) \subset L^2(\mathcal{A}_n, \tau)$. We have that

$$\begin{aligned} D_n x &= \sum_{k \geq 0} r^{(k-n)s} Q_k x = \sum_{k \geq 0} r^{(k-n)s} (I^{\otimes k} \otimes F \otimes E) x \\ &= \sum_{k=0}^{n-1} r^{(k-n)s} (I^{\otimes k} \otimes F \otimes E) x + \sum_{k \geq n} r^{(k-n)s} (I^{\otimes k} \otimes F \otimes E) x \\ &= \sum_{h=1}^{n} r^{-sh} (I^{\otimes (n-h)} \otimes F \otimes E) x + (I^{\otimes n} \otimes D_0) x. \end{aligned}$$

□

Corollary 2.5.5. *The Dirac operator D_∞ has the following form*

$$D_\infty = I_{-\infty,-1} \otimes D_0 + \sum_{k=1}^{\infty} r^{-sk} I_{-\infty,-k-1} \otimes F \otimes E, \tag{2.5.2}$$

where $I_{-\infty,k}$ is the identity on the factors with indices in $[-\infty, k]$.

Theorem 2.5.6. *Set $\mathcal{L} = \cup_n \mathcal{L}_n$, $\mathcal{M} = \mathcal{R} \otimes \mathcal{B}(L^2(\mathcal{A}_0, \tau))$. Then the triple $(\mathcal{L}, \mathcal{M}, L^2(\mathcal{R}, \tau) \otimes L^2(\mathcal{A}_0, \tau), D_\infty)$ is a finitely summable, semifinite, spectral triple, with Hausdorff dimension $\frac{2}{s}$. Moreover, the Dixmier trace τ_ω of $(D_\infty^2 + 1)^{-1/s}$ coincides with that of $(D_0^2 + 1)^{-1/s}$ (hence does not depend on ω) and may be written as:*

$$\tau_\omega((D_\infty^2 + 1)^{-1/s}) = \lim_{t \to \infty} \frac{1}{\log t} \int_0^t \left(\mu_{(D_0^2+1)^{-1/2}}(s) \right)^{\frac{2}{s}} ds.$$

Proof. By construction, $\mathcal{L}$ is a dense $*$-subalgebra of the C*-algebra $\mathcal{A}_\infty \subset \mathcal{M}$. Since $D_\infty = I_{-\infty,-1} \otimes D_0 + C$, where $C \in \mathcal{R} \otimes I$, D_∞ is affiliated to $\mathcal{M}$.

The thesis follows from Theorem 2.2.8 and what we have seen above. □

2.6 Inductive limits and the weak*-topology of their state spaces

First of all, we recall some definitions. Let $(\mathcal{L}, H, D)$ be a spectral triple over a unital C^*-algebra $\mathcal{A}$. Then we can define the following pseudometric on the state space

$$\rho_D(\phi,\psi) = \sup\{|\phi(x) - \psi(x)| : x \in \mathcal{A}, L_D(x) \le 1\}, \quad \phi,\psi \in S(\mathcal{A}),$$

where $L_D(x)$ is the seminorm $\|[D,x]\|$.

We have the following result proved by Rieffel.

Theorem 2.6.1. *(*[Rie98]*) The pseudo-metric ρ_D induces a topology equivalent to the weak*-topology if and only if the ball*

$$B_{L_D} := \{x \in \mathcal{A} : L_D(x) \le 1\}.$$

is totally bounded in the quotient space $\mathcal{A}/\mathbb{C}1$

If the above condition is satisfied, the seminorm L_D is said a *Lip-norm on $\mathcal{A}$*. In our examples we determined a semifinite spectral triple on $\mathcal{A}_\infty$. Our aim is to prove that the seminorm $L_{\widehat{D}_\infty}$, restricted to $\mathcal{A}_n$, is a Lip-norm equivalent to L_{D_n}, for any n, while it is not a Lip-norm on the whole inductive limit A_∞. Therefore, the pair $(\mathcal{A}_\infty, L_{\widehat{D}_\infty})$ is not a quantum compact metric space, whilst $\mathcal{A}_\infty$ is topologically compact (i.e. it is a unital C*-algebra).

Theorem 2.6.2. *Consider the Dirac operators $\widehat{D}_\infty$ determined in the previous sections. Then the sequence of the normic radii of the balls $B_{L_{D_n}}$ diverges. In particular, the seminorm $L_{\widehat{D}_\infty}$ on the inductive limit is not Lipschitz.*

Proof. Our aim is to show that $B_{L_{\widehat{D}_\infty}}$ is unbounded. Actually, we will exhibit a sequence in $B_{L_{\widehat{D}_\infty}}$ with constant seminorm and diverging quotient norm, which means that it is an unbounded set in $\varinjlim \mathcal{A}_k/\mathbb{C}$.

In the first place we consider the cases of the commutative and noncommutative torus. The noncommutative rational torus has centre isomorphic to the algebra of continuous functions on the torus. Thus, it is enough to exhibit a sequence only in the case of the torus.

Consider the following sequence

$$x_k = e^{2\pi i (A^k e_1, t)}$$

where $A := (B^T)^{-1}$.

Each $x_k \in C(\mathbb{T}_k) \subset \lim_i C(\mathbb{T}_i)$. We have that

$$\begin{aligned} \|[D_k, x_k]\| &= \|\sum_a \varepsilon^a [\partial_a, x_k]\| \leq \sum_a \|[\partial_a, x_k]\| \\ &= \sum_a \|\partial_a(x_k)\| = \sum_a \|2\pi i (A^k e_1, e_a) x_k\| \\ &\leq 2p\pi \|A^k e_1\| \leq 2p\pi \|A^k\| \to 0 \end{aligned}$$

where we used Proposition 2.2.6.

Consider the sequence $y_k := x_k / \|[D_k, x_k]\|$. This sequence has constant seminorm $L_{\widehat{\mathcal{D}}_\infty} \equiv L_{D_k}$. Since each element x_k has spectrum $\mathbb{T}$, then the quotient norm of x_k is equal to $\|x_k\|$ and thus the sequence $\{y_k\}$ is unbounded.

We now consider the case of the crossed products. With the same notations as above, consider the following sequence

$$x_k = U_{A^k e_1}$$

Each $x_k \in \mathcal{A}_k \subset \varinjlim \mathcal{A}_i$. We have that

$$\|[D_k, x_k]\| = \|[M_{\ell_k}, U_{A^k e_1}]\| \leq \sup_g |\ell_k(g) - \ell_k(g - A^k e_1)| \leq \|A^k e_1\| \leq \|A^k\| \to 0.$$

Since $\mathrm{sp}(x_k) = \mathbb{T}$, again the sequence $y_k := x_k / \|[D_k, x_k]\|$ has constant seminorm $L_{\widehat{\mathcal{D}}_\infty} \equiv L_{D_k}$ and increasing quotient norm.

Finally we take care of the UHF-algebra. Consider any matrix $b \in \big(M_r(\mathbb{C}) \setminus \mathbb{C}I\big) \subset UHF(r^\infty)$. We define the following sequence

$$x_n = I_{[-\infty, -n-1]} \otimes b \otimes I_{[-n+1, +\infty]},$$

where with the above symbol we mean that the matrix b is in the position $-n$ inside an infinite bilateral product where each factor is labelled by an integer. A quick computation shows that

$$[Q_k, x_n] = \begin{cases} 0 & \text{if } k > -n \\ \mathrm{id}_{-\infty, k-1} \otimes (b\mathrm{Tr}(\cdot) - \mathrm{Tr}(b\cdot)) \otimes \tau & \text{if } k = -n \\ \mathrm{id}_{-\infty, k-1} \otimes F \otimes \big(\bigotimes_{i=k+1}^{-n-1} \mathrm{Tr}(\cdot)\big) \otimes (\mathrm{Tr}(b\cdot) - b\mathrm{Tr}(\cdot)) \otimes \tau & \text{if } k < -n. \end{cases}$$

This means that $[D_\infty, x_n] = \sum_{k \leq -n} r^{ks} [Q_k, x_n]$.

We observe that each x_n has non-zero seminorm. In fact,

$$\begin{aligned} \|[D_\infty, x_n]\| &= \sup_{\|\xi\|=1} \|[D_\infty, x_n]\xi\| \\ &\geq \|[D_\infty, x_n] x_n^*\| \\ &= r^{-ns} \|\mathrm{Tr}(bb^*) - b\mathrm{Tr}(b^*)\| > 0 \end{aligned}$$

where in the last line we used that $[Q_k, x_n] x_n^* = 0$ for all $k \neq -n$. Moreover, we have that

$$\|[D_\infty, x_n]\| \leq 2\|b\| \left(\sum_{k \leq -n} r^{ks} \right) = 2\|b\| \frac{r^{s-ns}}{1 - r^s}$$

which tends to zero as n goes to infinity.

The sequence $y_k := x_k/\|[D_\infty, x_k]\|$ has bounded seminorm $L_{\widehat{D}_\infty}$ and increasing quotient norm.

We end this proof with an explanation of the second part of the statement of this Theorem. First of all, we observe that if the sequence of the normic radii of the balls $B_{L_{D_n}}$ diverges, then $B_{L_{\widehat{D}_\infty}}$ contains an unbounded subset with unbounded quotient norm. Therefore, since a compact subset is bounded, the ball $B_{L_{\widehat{D}_\infty}}$ cannot be compact. □

In a recent paper ([LP16]) Latrémolière and Packer studied the metric structure of noncommutative solenoids, namely of the inductive limits of quantum tori. In particular, they considered noncommutative tori as quantum compact metric spaces and proved that their inductive limits, seen as quantum compact metric spaces, are also limits in the sense of Gromov-Hausdorff propinquity (hence quantum Gromov-Hausdorff) of the inductive families. In our setting the inductive limit of the quantum tori is no longer a quantum compact metric space. The different result is a consequence of the different metric structure considered. Latrémolière and Packer described the inductive limit as a twisted group C^*-algebra on which there is an ergodic action of $G_\infty := \varprojlim \mathbb{T}$, and according to Rieffel ([Rie98]) a continuous length function on G_∞ gives rise to a Lip-seminorm. In our setting the seminorm may also be described in the same way, however the corresponding length function is unbounded, thus not continuous. We give an explicit description of this situation in a particular example.

Example 2.6.3. Consider the two-dimensional rational rotation algebra A_θ, with $\theta = 1/3$. With the former notation, set

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and define the morphism $\alpha : A_\theta \to A_\theta$ by $\alpha(U) = U^2$, $\alpha(V) = V^2$. Now we may consider the inductive limit $\varinjlim \mathcal{B}_n$ where $\mathcal{B}_n = A_\theta$ (see (2.0.1)). We observe that this case also fits in the setting of Latrémolière and Packer (see [LP16, Theorem 3.3]). Then, there exists a length function that induces the seminorm $L_{\widehat{D}_\infty}$.

Proof. Consider the standard length function on the circle $\ell(e^{2\pi i t}) := |t|$ for $t \in (-1/2, 1/2]$. There is an induced length function on $\mathbb{T}^2$, namely $\ell_0(z_1, z_2) := \max\{\ell(z_1), \ell(z_2)\}$. We define the following length function $\ell(g) := \sup_n 2^n \ell_0(g_n)$ on the direct product $\prod \mathbb{T}^2$, thus by restriction also on the projective limit $G_\infty := \varprojlim \mathbb{T}$ (with respect to the projection $\pi \equiv \alpha^* : \mathbb{T}^2 \to \mathbb{T}^2$, $\pi((z_1, z_2)) = (2z_1, 2z_2)$). For any $\phi \in \mathbb{R}^2$ we define the following action on A_θ: $\tilde{\rho}_\phi(f)(t) := f(t + 3\phi)$. Since $\theta = 1/3$, $\tilde{\rho}$ is the identity on A_θ when $\phi \in \mathbb{Z}^2$, hence there is an induced action of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ on A_θ. We denote this action with ρ. There is a naturally induced action ρ^∞ of the group $\prod_{i=0}^\infty \mathbb{T}^2$ on $\prod_{i=0}^\infty A_\theta$ given by $\rho_g^\infty(f_0, f_1, \dots) := (\rho_{g_0}(f_0), \rho_{g_1}(f_1), \dots)$ for any $g \in \prod_{i=0}^\infty \mathbb{T}^2$ and any $(f_0, f_1, \dots) \in \prod_{i=0}^\infty A_\theta$. We now check that the restriction of this action to G_∞ gives rise to an action on $\varinjlim A_\theta$. It is enough to prove the claim on the algebraic inductive limit alg$-\varinjlim A_\theta$. Let $(f_0, f_1, \ldots) \in$ alg$-\varinjlim A_\theta$. By definition there exists $n \in \mathbb{N}$ such that $f_{n+i} = \alpha^i(f_n)$ for all $i \in \mathbb{N}$. For any $g \in G_\infty$, we have that

$$\rho_{g_{n+i}}(f_{n+i}) = \rho_{g_{n+i}}(\alpha^i(f_n)) = \alpha^i(\rho_{\pi^n(g_{n+i})}(f_n)) = \alpha^i(\rho_{g_n}(f_n)).$$

For any $g \in G_\infty$ and any $X \in \varinjlim A_\theta$ we define the following seminorm

$$L_{\rho^\infty,\ell}(X) := \sup_{g\in G_\infty} \frac{\|\rho_g^\infty(X) - X\|}{\ell(g)}.$$

Any element $f_n \in \mathcal{B}_n$ embeds into $\varinjlim A_\theta$ as $X = (\underbrace{0, \ldots, 0}_{n}, f_n, \alpha(f_n), \alpha^2(f_n) \ldots)$. We have that

$$\begin{aligned}
L_{\rho^\infty,\ell}(X) &= \sup_{g\in G_\infty} \frac{\|\rho_g^\infty(X) - X\|}{\ell(g)} \\
&= \sup_{g\in G_\infty} \lim_i \sup \frac{\|\alpha^i(f_n)(z + 3g_{n+i}) - \alpha^i(f)(z)\|}{\ell(g)} \\
&= \sup_{g\in G_\infty} \frac{\|f_n(z + 3g_n) - f_n(z)\|}{\ell_0(g_n)} \frac{\ell_0(g_n)}{\ell(g)} \\
&= \left(\sup_{g_n} \frac{\|f_n(z + 3g_n) - f_n(z)\|}{\ell_0(g_n)}\right)\left(\sup_{g\in G_\infty} \frac{\ell_0(g_n)}{\ell(g)}\right) \\
&= \frac{L_0(f)}{2^n},
\end{aligned}$$

where the last two equalities hold because, for any $g_n \in \mathbb{T}^2$, we may find a sequence $g = \{g_i\}$ such that $\ell(g) = 2^n \ell_0(g_n)$ (if $g_n = e^{2\pi it}$ for $t \in (-1/2, 1/2]$ consider $g_{n+k} = e^{2\pi it/2^k}$) and L_0 is the Lipschitz seminorm $\sup_{h\in\mathbb{T}^2} \frac{\|f(z+h)-f(z)\|}{\ell_0(h)}$, which is equivalent to L_{D_0} (see [Rie98]). Denote by $\varphi_n : \mathcal{B}_n = A_\theta \to \mathcal{A}_n$ the natural isomorphism given by $\varphi_n(W(m,t)) := e^{2\pi i\theta(2^{-n}m,t)} W_0(2^n m)$ (cf. (2.3.4)) and consider the following seminorm on $\mathcal{B}_n$: $L_n(x) := L_D(\varphi_n(x)) = \|[D_n, \varphi_n(x)]\|$. Since the seminorm L_D is expressed in terms of the norm of some linear combinations of the two derivatives, one has that $L_n(x) = 2^{-n} L_0(x)$. Therefore, the former computation leads to $L_{\rho^\infty,\ell} = L_n$, when restricted to $\mathcal{B}_n$. □

2.7 Appendix: Some results in noncommutative integration theory

Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a f.n.s. trace, $T \hat{\in} \mathcal{M}$ a self-adjoint operator. We use the notation $e_T(\Omega)$ for the spectral projection of T relative to the measurable set $\Omega \subset \mathbb{R}$, and $\lambda_T(t) := \tau(e_{|T|}[t, +\infty))$, $\mu_T(t) := \inf\{s : \lambda_T(s) \le t\}$, for a τ-compact operator T.

Lemma 2.7.1. *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a f.n.s. trace, $T \widehat{\in} \mathcal{M}$ a self-adjoint operator, such that $\Lambda_T(s) := \tau(e_T(-s,s)) < \infty$ for any $s > 0$. Then*

(1) *$\Lambda_T(s) = \sup\{\tau(e) : \|Te\| < s, e \in Proj(\mathcal{M})\}$, $s > 0$,*

(2) *if $C \in \mathcal{M}_{sa}$, and $c := \|C\|$, then $\tau(e_{T+C}(-s,s)) < \infty$ for any $s \geq 0$, and $\Lambda_{T+C}(s) \leq \Lambda_T(s+c)$,*

(3) *if $e_T(\{0\}) = 0$, T^{-1} is τ-compact and $\Lambda_T(s) = \lambda_{|T|^{-1}}(s^{-1})$, $s > 0$.*

Proof. (1) Indeed,

$$a := \tau(e_T(-s,s)) = \sup\{\tau(e_T(-\sigma,\sigma)) : 0 \leq \sigma < s\} \leq \sup\{\tau(e) : \|Te\| < s\}.$$

Assume, by contradiction, there is $e \in \mathrm{Proj}(\mathcal{M})$ such that $\tau(e) > a$ and $\|Te\| < s$. For $\xi \in e\mathcal{H} \cap e_{|T|}[s,\infty)\mathcal{H}$, $\|\xi\| = 1$, we have $(\xi, T^*T\xi) < s^2$ and $(\xi, T^*T\xi) \geq s^2$, namely $e \wedge e_{|T|}[s,\infty) = \{0\}$. As a consequence,

$$e_{|T|}[s,\infty) = e_{|T|}[s,\infty) - e \wedge e_{|T|}[s,\infty) \sim e \vee e_{|T|}[s,\infty) - e \leq I - e$$

where $\sim$ stands for Murray - von Neumann equivalence. Passing to the orthogonal complements we get $a = \tau(e_T(-s,s)) \geq \tau(e) > a$, which is absurd.

(2) Set $\Omega_{T,s} = \{e \in \mathrm{Proj}(\mathcal{M}) : \|Te\| < s\}$; since $\|Te\| \leq \|(T+C)e\| + c$, we have that $\Omega_{T+C,s} \subseteq \Omega_{T,s+c}$, . The thesis follows from (1).

(3) A straightforward computation shows that $e_{|T|^{-1}}(s,+\infty) = e_T(-1/s, 1/s)$. Therefore T^{-1} is τ-compact [FK86] and the equality follows. □

Lemma 2.7.2. *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a f.n.s. trace, $T \widehat{\in} \mathcal{M}$ a positive self-adjoint operator T, with τ-compact resolvent, $d, t > 0$. Then, the following are equivalent*

(1) *exists $res_{s=d}\ \tau(T^{-s}e_T[t,+\infty)) = \alpha \in \mathbb{R}$,*

(2) *exists $res_{s=d}\ \tau((T^2+1)^{-s/2}) = \alpha \in \mathbb{R}$.*

Proof. Let us first observe that

$$\tau(T^{-s}e_T[t,+\infty)) = \int_t^\infty \lambda^{-s} d\tau(e_T(0,\lambda)), \tag{2.7.1}$$

$$\tau((T^2+1)^{-s/2}) = \int_0^\infty (\lambda^2+1)^{-s/2} d\tau(e_T(0,\lambda)), \tag{2.7.2}$$

and

$$(t^2+1)^{-s/2} \leq (\lambda^2+1)^{-s/2} \leq 1, \quad \forall \lambda \in [0,t],$$
$$t^s(1+t^2)^{-s/2}\lambda^{-s} \leq (\lambda^2+1)^{-s/2} \leq \lambda^{-s}, \quad \forall \lambda \in [t,+\infty),$$

therefore the finiteness of any of the two residues in the statement implies the finiteness of the two integrals (2.7.1), (2.7.2) above for any $s > d$. Then,

$$|\tau(T^{-s}e_T[t,+\infty)) - \tau((T^2+1)^{-s/2})| = \left|\int_t^\infty \lambda^{-s} d\tau(e_T(0,\lambda)) - \int_0^\infty (\lambda^2+1)^{-s/2} d\tau(e_T(0,\lambda))\right|$$
$$\leq \int_0^t (\lambda^2+1)^{-s/2} d\tau(e_T(0,\lambda)) + \frac{s}{2}\int_t^\infty \lambda^{-s-2} d\tau(e_T(0,\lambda)),$$

where the inequality follows by

$$\lambda^{-s} - (\lambda^2+1)^{-s/2} = \lambda^{-s}[1-(1+\frac{1}{\lambda^2})^{-s/2}] \leq \frac{s}{2}\lambda^{-s-2},$$

which, in turn, follows by

$$g(x) = 1-(1+x)^{-s/2} \leq \sup_{\xi\in[0,x]} g'(\xi)\, x = \frac{s}{2}x, \quad \text{for } x \geq 0$$

Finally, taking the limit for $s \to d^+$, we get

$$\lim_{s\to d^+} |\tau(T^{-s}e_T[t,+\infty)) - \tau((T^2+1)^{-s/2})| \leq \tau(e_T(0,t)) + \frac{d}{2}\int_t^\infty \lambda^{-(d+2)} d\tau(e_T(0,\lambda)) < \infty,$$

where the last integral is (2.7.1) with $s = d+2$, hence is finite, and we have proven the thesis. □

Lemma 2.7.3. *Let $(\mathcal{M},\tau)$ be a von Neumann algebra with a f.s.n. trace, T a self-adjoint operator affiliated with $\mathcal{M}$ with bounded compact inverse, $C \in \mathcal{M}_{sa}$ such that $T + C$ has bounded inverse. Then, the following are equivalent*

(1) *exists $res_{s=d}\ \tau(|T|^{-s}) = \alpha \in \mathbb{R}$,*

(2) *exists $res_{s=d}\ \tau(|T+C|^{-s}) = \alpha \in \mathbb{R}$.*

Proof. It is enough to prove that (1) $\implies$ (2). Set $c := \|C\|$. From Lemma 2.7.1, we get $\Lambda_{T+C}(s) \leq \Lambda_T(s+c)$ for every $s > 0$, hence $\lambda_{|T+C|^{-1}}(s) \leq \lambda_{|T|^{-1}}(\frac{s}{1+cs})$. Then, for $0 < \vartheta < 1$,

$$\begin{aligned}
\mu_{|T+C|^{-1}}(t) &= \inf\{s \geq 0 : \lambda_{|T+C|^{-1}}(s) \leq t\} \\
&\leq \inf\{s \geq 0 : \lambda_{|T|^{-1}}(\frac{s}{1+cs}) \leq t\} \\
&= \inf\{\frac{h}{1-ch} \geq 0 : \lambda_{|T|^{-1}}(h) \leq t\} \\
&= \inf\{\frac{h}{1-ch} : 0 \leq h < c^{-1}, \lambda_{|T|^{-1}}(h) \leq t\} \\
&\leq \inf\{\frac{h}{1-ch} : 0 \leq h \leq \vartheta c^{-1}, \lambda_{|T|^{-1}}(h) \leq t\} \\
&\leq (1-\vartheta)^{-1} \inf\{h : 0 \leq h \leq \vartheta c^{-1}, \lambda_{|T|^{-1}}(h) \leq t\} \\
&= \begin{cases} (1-\vartheta)^{-1}\inf\{h \geq 0 : \lambda_{|T|^{-1}}(h) \leq t\}, & \text{if } \lambda_{|T|^{-1}}(c^{-1}\vartheta) \leq t, \\ +\infty, & \text{otherwise,} \end{cases} \\
&= \begin{cases} (1-\vartheta)^{-1}\mu_{|T|^{-1}}(t), & \text{if } \lambda_{|T|^{-1}}(c^{-1}\vartheta) \leq t, \\ +\infty, & \text{otherwise.} \end{cases}
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\tau(|T+C|^{-s}) &= \int_0^\infty \mu_{|T+C|^{-1}}(t)^s\,dt \\
&\le \int_0^{\lambda_{|T|^{-1}}(c^{-1}\vartheta)} \mu_{|T+C|^{-1}}(t)^s\,dt + \int_{\lambda_{|T|^{-1}}(c^{-1}\vartheta)}^{+\infty} (1-\vartheta)^{-s}\mu_{|T|^{-1}}(t)^s\,dt \\
&= \int_0^{\lambda_{|T|^{-1}}(c^{-1}\vartheta)} \left(\mu_{|T+C|^{-1}}(t)^s - (1-\vartheta)^{-s}\mu_{|T|^{-1}}(t)^s\right)dt + (1-\vartheta)^{-s}\tau(|T|^{-s}) \\
&\le \left(\|(T+C)^{-1}\|^s + (1-\vartheta)^{-s}\|T^{-1}\|^s\right)\lambda_{T^{-1}}(c^{-1}\vartheta) + (1-\vartheta)^{-s}\tau(|T|^{-s}) < \infty.
\end{aligned}$$

Passing to the residues, we get $\limsup_{s\to d^+}(s-d)\tau(|T+C|^{-s}) \le (1-\vartheta)^{-d}\mathrm{res}_{s=d}\,\tau(|T|^{-s})$, hence, by the arbitrariness of ϑ, $\limsup_{s\to d^+}(s-p)\tau(|T+C|^{-s}) \le \mathrm{res}_{s=d}\,\tau(|T|^{-s})$. Exchanging T with $T+C$ we get $\mathrm{res}_{s=d}\,\tau(|T|^{-s}) \le \liminf_{s\to d_+}(s-p)\tau(|T+C|^{-s})$, hence the thesis. □

Proposition 2.7.4. *Let $(\mathcal{M},\tau)$ be a von Neumann algebra with a f.n.s. trace, T a self-adjoint operator affiliated with $\mathcal{M}$ with compact resolvent, $C \in \mathcal{M}_{sa}$. Then, the following are equivalent*

(1) *exists* $\mathrm{res}_{s=d}\,\tau((T^2+1)^{-s/2}) = \alpha \in \mathbb{R}$,

(2) *exists* $\mathrm{res}_{s=d}\,\tau((T+C)^2+1)^{-s/2}) = \alpha \in \mathbb{R}$.
In particular, the abscissas of convergence coincide.

Proof. By Lemma 2.7.2, the thesis may be rewritten as

$$\exists\ \mathrm{res}_{s=d}\,\tau(|T|^{-s}e_{|T|}[t,+\infty)) = \alpha \in \mathbb{R} \iff \exists\ \mathrm{res}_{s=d}\,\tau(|T+C|^{-s}e_{|T+C|}[t,+\infty)) = \alpha.$$

Since the operator

$$\begin{aligned}
C' :&= (T+C)e_{|T+C|}[t,+\infty) - Te_{|T|}[t,+\infty) \\
&= (T+C)e_{|T+C|}[0,+\infty) - Te_{|T|}[0,+\infty) - (T+C)e_{|T+C|}[0,t) + Te_{|T|}[0,t) \\
&= C - (T+C)e_{|T+C|}[0,t) + Te_{|T|}[0,t)
\end{aligned}$$

is bounded and self-adjoint, we may apply Lemma 2.7.3 to the operators $(T+C)e_{|T+C|}[t,+\infty)$ and $Te_{|T|}[t,+\infty)$, proving the Proposition. □

Chapter 3

The inner structure of $\mathcal{Q}_2$ and its automorphism group

The objectives of this chapter are to describe some aspects of the inner structure of the 2-adic ring C^*-algebra, study in detail some remarkable groups of automorphisms, and investigate the relation between on $\mathcal{O}_2$ and $\mathcal{Q}_2$. In particular, for the first point we want to prove that some notable commutative C^*-algebras are actually maximal abelian in $\mathcal{Q}_2$ and that the C^*-algebra generated by the isometry S_2 has trivial relative commutant. The understanding of these features will allow us to explicitely describe some families of automorphisms, in particular those fixing $C^*(U)$. We will also consider the problem of extending automorphisms of $\mathcal{O}_2$ to $\mathcal{Q}_2$. This chapter is based on the results contained in the paper [ACR16].

3.1 Some preliminary results

3.1.1 Some extensible endomorphisms

In the first chapter, we observed that $\mathcal{Q}_2$ contains a distinguished copy of $\mathcal{O}_2$, namely the C*-algebra generated by S_2 and $S_1 := US_2$. We begin this section by exhibiting some endomorphisms of $\mathcal{Q}_2$ that are extensions of endomorphisms of $\mathcal{O}_2$, they will play a role in the sequel. We anticipate that the existence of these endomorphisms will depend on the universality of the algebra $\mathcal{Q}_2$.

The first endomorphism is the canonical shift which is defined on every $x \in \mathcal{O}_2$ as $\varphi(x) = S_1 x S_1^* + S_2 x S_2^*$. This endomorphism may be extended to $\mathcal{Q}_2$ by setting

$$\tilde{\varphi}(x) = US_2 x S_2^* U^* + S_2 x S_2^* \quad \text{for any } x \in \mathcal{Q}_2 \ .$$

We observe that intertwining rules $S_i x = \tilde{\varphi}(x) S_i$ for any $x \in \mathcal{Q}_2$ with $i = 1, 2$ still hold true. Moreover, since $S_2 U = U^2 S_2$ and $S_1 U = U^2 S_1$, we have that

$$\tilde{\varphi}(U) = S_1 U S_1^* + S_2 U S_2^* = U^2 S_1 S_1^* + U^2 S_2 S_2^* = U^2$$

where we used the fact that $S_1 S_1^* + S_2 S_2^* = 1$. We make two last remarks. The continuous functional calculus of a normal operator commutes with any endomorphism, thus we also have that $\tilde{\varphi}(f(U)) = f(U^2)$ for any continuous function f on $\mathbb{T}$ ([Dix77, Proposition 1.5.3, p.14]). The same identity also holds true with any Borel function whenever $\mathcal{Q}_2$ is represented on some Hilbert space.

The second example is given by the so-called flip-flop automorphism, namely the automorphism of $\mathcal{O}_2$ defined as $\lambda_f(S_2) := S_1$, $\lambda_f(S_1) := S_2$. In fact, set $U' := U^*$ and $S_2' := US_2$. Then we have that

$$\begin{aligned} &S_2'S_2'^* + U'S_2'S_2'^*U^* = S_1S_1^* + U^*S_1S_1^*U = S_1S_1^* + S_2S_2^* = 1 \\ &S_2'U' = S_1U^* = U^{*2}S_1 = U'^2S_2' \end{aligned}$$

where we used that $S_2U^* = U^{*2}S_2$ (which may be easily proved in the canonical expectation). By the universality of $\mathcal{Q}_2$, there exists a unique endomorphism $\tilde{\lambda}_f \in \mathrm{End}(\mathcal{Q}_2)$ such that $\tilde{\lambda}_f(U) = U' = U^*$ and $\tilde{\lambda}_f(S_2) = S_2' = US_2$. As already mentioned in Section 1.5, the 2-adic ring C^*-algebra is simple, thus this endomorphism is necessarily injective. The endomorphism is actually surjective since $\tilde{\lambda}_f(U^*) = U$, $\tilde{\lambda}_f(US_2) = S_2$. In particular, the range of this endomorphism actually contains the whole $\mathcal{Q}_2$. The automorphism $\tilde{\lambda}_f$ will be called the flip-flop automorphism of $\mathcal{Q}_2$.

The third and final example, is given by the gauge automorphisms. The torus $\mathbb{T}$ acts on the Cuntz algebra $\mathcal{O}_2$ through the gauge automorphisms α_θ given by $\alpha_\theta(S_i) := e^{i\theta}S_i$, where θ is any real number. These automorphisms extend to automorphisms of $\mathcal{Q}_2$ by setting $\alpha_\theta(U) := U$. In fact, if we define $U' := U$ and $S_2' := e^{i\theta}S_2$, we may see that

$$\begin{aligned} &S_2'S_2'^* + U'S_2'S_2'^*U^* = e^{i\theta}e^{-i\theta}S_2S_2^* + e^{i\theta}e^{-i\theta}US_2S_2^*U^* = S_2S_2^* + US_2S_2^*U^* = 1 \\ &U'^2S_2' = e^{i\theta}U^2S_2 = e^{i\theta}S_2 = S_2'U' \ . \end{aligned}$$

Thus, again by using the universality of $\mathcal{Q}_2$, for each θ there exists an automorphism $\tilde{\alpha}_\theta \in \mathrm{Aut}(\mathcal{Q}_2)$ such that $\tilde{\alpha}_\theta(U) = U$ and $\tilde{\alpha}_\theta(S_2) = e^{i\theta}S_2$. With a slight abuse of terminology, the automorphisms $\tilde{\alpha}_\theta$ obtained above will be referred to as the gauge automorphisms. To conclude, it is worth noting that the flip-flop and the gauge automorphisms commute.

3.1.2 The gauge-invariant subalgebra

The subalgebra of $\mathcal{O}_2$ consisting of elements invariant under all the gauge automorphism, the so-called gauge-invariant subalgebra of $\mathcal{O}_2$, is given by $\mathcal{F}_2$ (see [Dav96, Theorem V.4.3, p.145]). This algebra is known to be isomorphic to a remarkable C^*-algebra: the CAR algebra. Now we want to describe the corresponding gauge-invariant subalgebra of $\mathcal{Q}_2$, which will be denoted by $\mathcal{Q}_2^{\mathbb{T}}$, as the closure of a suitable linear span. We start with some preliminary and useful result. We recall that in Section 1.5 (and as in [Cun77]) we denoted by W_2 the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \in \{1,2\}$ and $n \in \mathbb{N}$.

Proposition 3.1.1. *$\mathcal{Q}_2 = \overline{\mathrm{span}}\{S_\mu S_\nu^* U^k : \mu, \nu \in W_2, k \in \mathbb{Z}\}$.*

Proof. In order to prove the equality above all we have to do is observe that the following relations allow us to take both U and U^* from the left to the right side of any monomial of the form $S_\mu S_\nu^*$.

- $US_1 = S_2U$
- $US_2 = S_1$
- $US_1^* = S_2^*U$
- $US_2^* = S_2^*U^2$
- $U^*S_1 = S_2$
- $U^*S_2 = S_1U^*$
- $U^*S_1^* = S_1^*(U^*)^2$
- $U^*S_2^* = S_1^*U^*$

The above relations can be easily checked with some manipulations of the commutation relation $U^2S_2 = S_2U$, the definition of the isometry $S_1 = US_2$, and/or by using the canonical representation. □

The gauge automorphisms yield a conditional expectation $\widetilde{E} : \mathcal{Q}_2 \to \mathcal{Q}_2^{\mathbb{T}}$

$$\widetilde{E}(x) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{\alpha}_\theta(x)\mathrm{d}\theta \quad x \in \mathcal{Q}_2 \ .$$

Now since $\widetilde{E}(S_\mu S_\nu^* U^k) = S_\mu S_\nu^* U^k \int_0^{2\pi} e^{i(\ell(\mu)-\ell(\nu))\theta}\mathrm{d}\theta$, we also have $\widetilde{E}(S_\mu S_\nu^* U^k) = 0$ if and only if $\ell(\mu) \neq \ell(\nu)$. This helps to prove the following result.

Proposition 3.1.2. *The equalities below hold:*

$$\mathcal{Q}_2^{\mathbb{T}} = \overline{\mathrm{span}}\left\{S_\mu S_\nu^* U^k : \mu,\nu \in W_2, \ell(\mu) = \ell(\nu), k \in \mathbb{Z}\right\} = C^*(U, \mathcal{F}_2) \subset \mathcal{Q}_2 \ .$$

Proof. The second equality is obvious. The inclusion $\mathcal{Q}_2^{\mathbb{T}} \supset \overline{\mathrm{span}}\{S_\mu S_\nu^* U^k : \ell(\mu) = \ell(\nu)\}$ is clear by the former remarks on the conditional expectation. So we only have to prove the inclusion $\mathcal{Q}_2^{\mathbb{T}} \subset \overline{\mathrm{span}}\{S_\mu S_\nu^* U^k : \ell(\mu) = \ell(\nu)\}$. If $x \in \mathcal{Q}_2^{\mathbb{T}}$, then $x = \widetilde{E}(x)$. Now pick a sequence $\{x_n\}$ in the algebraic linear span of the set $\{S_\mu S_\nu^* U^k : \mu,\nu \in W_2, k \in \mathbb{Z}\}$ such that $\|x_n - x\|$ tends to zero. As $\tilde{E}$ is a continuous map, $\|\widetilde{E}(x_n) - \widetilde{E}(x)\| = \|\widetilde{E}(x_n) - x\|$ tends to zero as well. The conclusion follows from the fact that $\widetilde{E}(x_n) \in \mathrm{span}\{S_\mu S_\nu^* U^k : \ell(\mu) = \ell(\nu)\}$ by the remark we made above. □

It is well worth pointing out that $C^*(\mathcal{F}_2, U) = C^*(\mathcal{D}_2, U)$ is the Bunce-Deddens algebra of type 2^∞, see [BOS16, Remark 2.8].

3.1.3 The canonical representation

In this section we state and prove some fundamental properties of the canonical representation. This representation of $\mathcal{Q}_2$ was first exhibited in [LL12].

Although we already defined it in Section 1.5, we recall that this representation acts on $\ell_2(\mathbb{Z})$ through the operators $S_2, U \in \mathcal{B}(\ell_2(\mathbb{Z}))$ given by $S_2 e_k := e_{2k}$ and $Ue_k := e_{k+1}$, where $\{e_k : k \in \mathbb{Z}\}$ is the canonical orthonormal basis of $\ell_2(\mathbb{Z})$, i.e. $e_k(m) = \delta_{k,m}$. We observe that 1 is the only eigenvalue of S_2, with eigenspace $\mathbb{C}e_0$. Similarly, S_1 has only one eigenvalue, namely 1, whose eigenspace is $\mathbb{C}e_{-1}$.

Proposition 3.1.3. *The canonical representation of $\mathcal{Q}_2$ is irreducible.*

Proof. Let $M \subset \ell_2(\mathbb{Z})$ be a $\mathcal{Q}_2$-invariant closed subspace. If P is the associated orthogonal projection, then $P \in \mathcal{Q}_2'$. In particular, $S_2 P = PS_2$, and so $S_2 Pe_0 = Pe_0$. As the eigenspace of S_2 corresponding to the eigenvalue 1 is spanned by e_0, we must have either $Pe_0 = e_0$ or $Pe_0 = 0$. In the first case, $e_0 \in M$, and therefore $C^*(U)e_0 \subset M$, which says that $M = \ell_2(\mathbb{Z})$ because e_0 is a cyclic vector for $C^*(U)$. In the second, $e_0 \in M^\perp$ instead. As above, $M^\perp$ being $\mathcal{Q}_2$-invariant too, we have $M^\perp = \ell_2(\mathbb{Z})$, i.e. $M = 0$. Note, however, that $\mathcal{O}_2$ does not act irreducibly on $\ell_2(\mathbb{Z})$, for the closed span of the set $\{e_k : k = 0, 1, \ldots, \}$ is obviously a proper $\mathcal{O}_2$-invariant subspace. $\square$

Define $\mathcal{H}_+$ and $\mathcal{H}_-$ as the closed subspaces of $\ell_2(\mathbb{Z})$ given by

$$\mathcal{H}_+ := \overline{\mathrm{span}\{e_k : k \geq 0\}}$$
$$\mathcal{H}_- := \overline{\mathrm{span}\{e_k : k < 0\}}\,.$$

The canonical representation restricts to $\mathcal{O}_2$ as a reducible representation, which we denote by π. In particular, it is a direct sum of two inequivalent irreducible representations of $\mathcal{O}_2$, namely the restriction of π to $\mathcal{H}_\pm$.

Proposition 3.1.4. *The subspaces $\mathcal{H}_\pm$ are both $\mathcal{O}_2$-irreducible.*

Proof. We only need to worry about $\mathcal{H}_+$, for $\mathcal{H}_-$ is dealt with in much the same way. Exactly as above, if $M \subset H_+$ is an $\mathcal{O}_2$-invariant subspace, then either M or its orthogonal complement $M^\perp$ must contain e_0. The proof is thus complete if we can show that an $\mathcal{O}_2$-invariant subspace containing e_0, say N, is the whole $\mathcal{H}_+$, and this is proved once we show $e_k \in N$ for every $k \geq 0$. This is in turn easily achieved by induction on k. Suppose we have proved $\{e_l : l = 0, 1, \cdots, k\} \subset N$. For the inductive step we have two cases, according as $k+1$ is even or odd. If it is even, then $e_{k+1} = S_2 e_{\frac{k+1}{2}}$; if it is odd, then $e_{k+1} = S_1 e_{\frac{k}{2}}$. In either cases we see that e_{k+1} is in N, as wished. $\square$

Denoting by $\pi_\pm$ the restriction of π to $\mathcal{H}_\pm$ respectively, the decomposition into irreducible representations $\pi = \pi_+ \oplus \pi_-$ has just been proved to hold.

Lemma 3.1.5. *If π_+ and π_- are the irreducible representations defined above, then $\pi_+ \not\sim \pi_-$.*

Proof. It is enough to note that $\pi_+(S_2)$ has 1 in its point spectrum, whereas $\pi_-(S_2)$ does not. $\square$

We will denote by $E_\pm$ the orthogonal projections onto $\mathcal{H}_\pm$ respectively. We observe that the basis vectors e_k are all cyclic and separating for U. Therefore the W^*-algebra generated by U is a maximal abelian von Neumann algebra of $B(\ell_2(\mathbb{Z}))$. More precisely, we have the following proposition.

Proposition 3.1.6. *$W^*(U) \subset B(\ell_2(\mathbb{Z}))$ is a* MASA *isomorphic with $L^\infty(\mathbb{T})$ with respect to the Haar measure of $\mathbb{T}$.*

Proof. It is a well-known fact that $\sigma(U) = \mathbb{T}$. Accordingly, we shall only prove that the spectral measure associated with e_0 is just the Haar measure of $\mathbb{T}$. If $p(z)$ is a Laurent polynomial, say $p(z) = \sum_{k=-n}^n a_k z^k$, then we have $\int_{\mathbb{T}} p(z) d\mu(z) = (p(U)e_0, e_0) = \sum_{k=-n}^n a_k (U^k e_0, e_0) = \sum_{k=-n}^n a_k (e_k, e_0) = a_0 = \mathrm{Res}_{z=0} \frac{p(z)}{z} = \frac{1}{2\pi i} \int_\gamma \frac{p(z)}{z} dz = \frac{1}{2\pi} \int_0^{2\pi} p(e^{i\theta}) d\theta$. $\square$

We also take this opportunity to exploit the canonical representation to show that $\mathcal{D}_2'' \in \mathcal{B}(\ell_2(\mathbb{Z}))$ is a maximal abelian subalgebra as well. This will in turn be vital to conclude that $\mathcal{D}_2$ is a maximal abelian subalgebra of $\mathcal{Q}_2$. Henceforward we shall denote by $\ell_\infty(\mathbb{Z})$ the atomic *MASA* of $\mathcal{B}(\ell_2(\mathbb{Z}))$ acting through diagonal operators with respect to the canonical basis.

Proposition 3.1.7. *In the canonical representation we have $\mathcal{D}_2' = \ell_\infty(\mathbb{Z})$.*

Proof. Since $\ell_\infty(\mathbb{Z})$ is a MASA, it is enough to prove that $\mathcal{D}_2'' = \ell_\infty(\mathbb{Z})$, which will be immediately checked once we have proved that the projections E_k onto $\mathbb{C}e_k$ all belong to the strong closure of $\mathcal{D}_2$. To begin with, we note that the sequence $\{S_2^n (S_2^*)^n\} \subset \mathcal{D}_2$ strongly converges to E_0. But then the sequence $\{U^k S_2^n (S_2^*)^n U^{-k} : n \in \mathbb{N}\}$ strongly converges to E_k. The conclusion now follows from the fact that $\mathcal{D}_2$ is globally invariant under $\mathrm{Ad}(U)$. $\square$

3.2 Structure results

3.2.1 Two maximal abelian subalgebras

The goal of the present section is show that both $C^*(U)$ and $\mathcal{D}_2$ are maximal abelian subalgebras of $\mathcal{Q}_2$. These facts will yield many consequences that we will describe in the following sections.

We start with $C^*(U)$. We will need the conditional expectations from $\mathcal{B}(\mathcal{H})$ onto a maximal subalgebra introduced in the classic work of Kadison and Singer [KS], to which the interested reader is referred although we do outline their construction. More precisely, we next show as a key lemma to achieve our result that for any proper (see again [KS] for the terminology) conditional expectation E from $\mathcal{B}(\mathcal{H})$ onto $W^*(U)$ we have that $E[S_\alpha S_\beta^*]$ is at worst a monomial in U. To this aim, the needed computations are far more easily made in the unitarily equivalent irreducible representation obtained out of the canonical representation via Fourier transform. In this representation $\mathcal{Q}_2$ acts on $L^2(\mathbb{T})$ (with respect to the Haar measure of $\mathbb{T}$) and the operators S_2, U take the forms $(S_2 f)(z) = f(z^2)$ and $(Uf)(z) = zf(z)$. The isometry S_1 is accordingly given by $(S_1 f)(z) = zf(z^2)$. As for the adjoints, they are immediately recognized to be given by $(S_2^* f)(z) = \frac{1}{2}\sum_{w^2=z} f(w)$ and $(S_1^* f)(z) = \frac{1}{2}\sum_{w^2=z} \bar{w} f(w)$. Note that in this representation $W^*(U)$ is nothing but $L^\infty(\mathbb{T})$ acting on $L^2(\mathbb{T})$ by multiplication. We can now go on to prove one of the two main results of this section. Our proof is in turn divided into a series of preliminary lemmata. Before we enter into the details of the argument, however, we rather quickly recall how conditional expectations form $\mathcal{B}(H)$ onto $W^*(U)$ can be obtained. We keep the same notations as in the aforementioned paper as to allow the reader to refer to it effortlessly whenever necessary. For a given $T \in \mathcal{B}(\mathcal{H})$, we set $T^{|P} := PTP + (I - P)T(I - P)$ for any projection P in $W^*(U)$. If $\{P_i\}$ is a generating sequence of projections of $W^*(U)$, then every cluster point of the sequence $\{T^{|P_1|P_2|\ldots|P_n}\}$ lies in $W^*(U)' = W^*(U)$, as explained in [KS]. This enables us to define a conditional expectation from $B(\mathcal{H})$ onto $W^*(U)$ associated with each ultrafilter $p \in \beta\mathbb{N}$, which in that paper is denoted by $\mathcal{D}_p$, simply by taking the strong limit of the subnet corresponding to that ultrafilter. We now take the sequence $P_{n,k} \in W^*(U)$ of the projections corresponding to the Borel sets $\delta_{n,k} := \left\{e^{2\pi i x} : \frac{k-1}{n} < x \leq \frac{k}{n}\right\}$ for every $n \in \mathbb{N}$ and any $k = 1, \ldots, n$, namely $(P_{n,k} f)(z) = \chi_{\delta_{n,k}}(z) f(z)$. Finally, we denote by E any of the conditional expectations that come from the construction sketched above (in the end we will show that there exists a unique conditional expectation from $\mathcal{Q}_2$ onto $C^*(U)$). As we will see, actually we the only necessary ingredients will be the existence of a conditional expectation (due to Kadison and Singer) and the general properties of conditional expectations.

Lemma 3.2.1. *With the notations above, for every non-zero $k \in \mathbb{N} \setminus \{0\}$ we have $E[S_2^k] = E[(S_2^*)^k] = 0$.*

Proof. By using the fact that $E[\cdot]$ is a $*$-linear map it is enough to prove that $E[S_2^k] = 0$. We prove the case $k = 1$, the other cases are dealt with the same reasoning. By using $W^*(U)$-linearity we have that

$$\begin{aligned}
E[S_2 U] &= E[U^2 S_2] \\
E[S_2]U &= U^2 E[S_2] \\
E[S_2]U - E[S_2]U^2 &= 0 \\
E[S_2](U - U^2) &= 0
\end{aligned}$$

since $E[S_2] = f(U) \in L^\infty(\mathbb{T})$, the above equality may be written as

$$f(z)(z - z^2) = 0$$

which implies $f(z) = 0$ almost everywhere since the polynomial $z - z^2$ has only two zeroes, namely 0, 1. $\square$

Lemma 3.2.2. *We have that* $E[S_2^k(S_2^*)^k] = 2^{-k}$ *for every non-negative integer* k.

Proof. To begin with, we observe that

$$E[S_\alpha S_\alpha^*] = U^h E[S_2^{|\alpha|}(S_2^*)^{|\alpha|}]U^{-h} = E[S_2^{|\alpha|}(S_2^*)^{|\alpha|}],$$

for some positive integer h. Since $1 = \sum_{|\alpha|=k} S_\alpha S_\alpha^*$ we have that

$$1 = \sum_{|\alpha|=k} E[S_\alpha S_\alpha^*] = 2^k E[S_2^{|\alpha|}(S_2^*)^{|\alpha|}].$$

This implies that $E[S_2^k(S_2^*)^k] = 2^{-k}$. □

Lemma 3.2.3. *We have that* $E[S_2^k(S_2^*)^m] = 0$ *for* $k, m \neq 0$, $k \neq m$.

Proof. It is not restrictive to suppose that $k < m$. Set $x := E[S_2^k(S_2^*)^m]$. By using that $1 = \sum_{|\alpha|=k} S_\alpha S_\alpha^*$, we get that

$$\begin{aligned}
x &= E[S_2^k(S_2^*)^m] = \\
&= -\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} E[S_\alpha^k(S_\alpha^*)^k(S_2^*)^{m-k}] + E[(S_2^*)^{m-k}] = \\
&= -\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} E[S_\alpha^k(S_\alpha^*)^k(S_2^*)^{m-k}] = \\
&= -\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} E[U^{h(\alpha)}S_2^k(S_\alpha^*)^k(S_2^*)^{m-k}] = \\
&= -\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} U^{h(\alpha)}E[S_2^k(S_2^*)^k U^{-h(\alpha)}(S_2^*)^{m-k}] = \\
&= -\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} U^{h(\alpha)}E[S_2^k(S_2^*)^k(S_2^*)^{m-k}U^{-2(m-k)h(\alpha)}] = \\
&= -\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} U^{h(\alpha)}E[S_2^k(S_2^*)^k(S_2^*)^{m-k}]U^{-2(m-k)h(\alpha)} = \\
&= -\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} E[S_2^k(S_2^*)^m]U^{-2(m-k-1)h(\alpha)} \\
&= -\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} xU^{-2(m-k-1)h(\alpha)} \\
&= -x\sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} U^{-2(m-k-1)h(\alpha)}
\end{aligned}$$

where $h(\alpha)$ is an integer-valued function. We stress that all we used is Lemma 3.2.1, the relation $S_2U = U^2S_2$ and $W^*(U)$-linearity of E. From the above computation, along with $W^*(U)$ being commutative, we find that

$$x\left(1 - \sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} U^{-2(m-k-1)h(\alpha)}\right) = \left(1 - \sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} U^{-2(m-k-1)h(\alpha)}\right)x = 0.$$

Therefore, in the canonical representation we see that $M_p x = xM_p = 0$, where p is the Laurent polynomial $p(z) := 1 - \sum_{|\alpha|=k,\alpha\neq(2,\dots,2)} z^{-2(m-k-1)h(\alpha)}$. This means that the range of x must be contained in $\ker(M_{p(z)})$. But because the multiplication operator M_p is injective, being the zero set of p a null set, the identity $M_p x = 0$ implies that $x = 0$, as desired. □

With the above result we have now made it plain that $E[S_\alpha S_\beta^*]$ is always a monomial in U, when it is not a scalar. Thus we have at last all the necessary information to complete our proof of $C^*(U)$ being a maximal abelian algebra.

Theorem 3.2.4. *$C^*(U)$ is a maximal abelian C^*-subalgebra of $\mathcal{Q}_2$.*

Proof. As $C^*(U)' \cap \mathcal{Q}_2 = W^*(U) \cap \mathcal{Q}_2$, it is enough to prove that given $f \in L^\infty(\mathbb{T})$ with $f(U)$ in $\mathcal{Q}_2$, then f is in fact a continuous function. Now if $f(U)$ belongs to $\mathcal{Q}_2$, it is also the norm limit of a sequence $\{x_k\} \subset \mathcal{Q}_2$ with each x_k taking the form $\sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h$. If $E : B(\mathcal{H}) \to W^*(U)$ is any of the proper conditional expectations considered above, we have $f(U) = E(f(U)) = \lim_k E(x_k)$. But then each $E(x_k)$ is of the form $\sum_{\alpha,\beta,h} c_{\alpha,\beta,h} U^{h+k_{\alpha,\beta}}$, where $U^{k_{\alpha,\beta}}$ is nothing but $E(S_\alpha S_\beta^*)$. Consequently, there exists a sequence of Laurent polynomials p_k such that $\|f(U) - p_k(U)\|$ tends to zero, that is $f(U) \in C^*(U)$, as maintained. □

As a byproduct of the above result, we obtained a conditional expectation from $E : \mathcal{Q}_2 \to C^*(U)$, which is obtained by restricting any of the aforesaid conditional expectations to $\mathcal{Q}_2$. Now we want to show that it is the sole conditional $\mathcal{Q}_2$ onto $C^*(U)$ and that it is indeed faithful. We observe that $W^*(U)$ which has great many conditional expectations, cf. [KS]. On the other hand, the above computations show that every conditional expectation F from $\mathcal{Q}_2$ onto $C^*(U)$ such that $F(S_2^k) = 0$ for every natural number k must coincide with E. We next show that E is faithful. This is actually a straightforward consequence of a general well-known result due to Tomiyama [Tom72], whose proof in our setting is nevertheless included for the sake of completeness, being utterly independent of Tomiyama's work to boot.

Proposition 3.2.5. *The conditional expectations from $\mathcal{B}(\mathcal{H})$ onto a MASA defined by Kadison and Singer are faithful.*

Proof. If T is an α-coercive operator, i.e. $(Tx, x) \geq \alpha\|x\|^2$ with $\alpha > 0$, then $T^{|P}$ is α-coercive as well regardless of the projection P. In particular, if $T \in B(\mathcal{H})$ is a coercive operator, then $E[T]$ cannot zero, being by definition a weak limit of coercive positive operators all with the same constant as T. If now T is any non-zero positive operator and $\varepsilon > 0$ is any real number with $\varepsilon < \|T\|$, the spectral theorem provides us with an orthogonal decomposition $\mathcal{H} = M_\varepsilon \oplus N_\varepsilon$ with M_ε and N_ϵ both T-invariant and such that the restriction $T\restriction_{N_\varepsilon}$ is ε-coercive. The remark we started our proof with allows to conclude that $E[T]$ is not zero either. □

More importantly, the conditional expectation exhibited above is indeed unique.

Theorem 3.2.6. *The conditional expectation $E : \mathcal{Q}_2 \to C^*(U)$ is unique.*

Proof. Let F be a conditional expectation from $\mathcal{Q}_2$ onto $C^*(U)$. The commutation rules $S_2^k U = U^{2^k} S_2^k$ give $F[S_2^k]U = U^{2^k} F[S_2^k] = F[S_2^k]U^{2^k}$. If we now set $f(U) := F[S_2^k]$, we see that $f(z)(z - z^{2^k}) = 0$, hence $f(z) = 0$ for every $z \in \mathbb{T}$. This says $F[S_2^k] = 0$, i.e. $F = E$. □

We can now move on to $\mathcal{D}_2$. The plan of the proof is the same as before. We will again use the conditional expectation defined by Kadison and Singer. This result can be seen as a generalization of the well-known property of $\mathcal{D}_2$ being maximal in $\mathcal{O}_2$. We start with the following some lemmas. We still denote by E the unique faithful conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\ell_\infty(\mathbb{Z})$. As known, this is simply given by $(E[T]e_i, e_j) = (Te_i, e_j)\delta_{i,j}$.

Lemma 3.2.7. *The following relations hold:*

- $E[U^k] = \delta_{k,0} I$,
- $E[S_\alpha S_\alpha^*] = S_\alpha S_\alpha^*$,
- *if* $|\alpha| \neq |\beta|$, $E[S_\alpha S_\beta^* U^h]$ *is either* 0 *or* E_i,
- *if* $|\alpha| = |\beta|$, $E[S_\alpha S_\beta^* U^h]$ *is either* 0 *or* $S_\alpha S_\beta^* U^h$.

Proof. The first two equalities need no proof. For the third, without harming generality, we may suppose that $|\alpha| < |\beta|$

$$\begin{aligned} E[S_\alpha S_\beta^* U^h] &= E[U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\beta|}U^{h-h(\beta)}] \\ &= E[U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\alpha|}U^{-h(\alpha)}U^{h(\alpha)}(S_2^*)^{|\beta|-|\alpha|}U^{h-h(\beta)}] \\ &= U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\alpha|}U^{-h(\alpha)}E[U^{h(\alpha)}(S_2^*)^{|\beta|-|\alpha|}U^{h-h(\beta)}]. \end{aligned}$$

where $h(\alpha)$ and $h(\beta)$ are positive integers. Accordingly, we are led to compute $E[U^h(S_2)^k U^l]$, where $h \in \mathbb{Z}$. Now the condition $U^h(S_2)^k U^l e_i = e_i$ implies that $i = 2^k(i + l) + h$, and because the former equation has a unique solution, we get the thesis. Finally, for the fourth we have that

$$\begin{aligned} E[S_\alpha S_\beta^* U^h] &= E[U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\beta|}U^{h-h(\beta)}] \\ &= E[U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\alpha|}U^{-h(\alpha)}U^{h-h(\beta)+h(\alpha)}] \\ &= U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\alpha|}U^{-h(\alpha)}E[U^{h-h(\beta)+h(\alpha)}] \\ &= \delta_{h-h(\beta)+h(\alpha),0}U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\alpha|}U^{-h(\alpha)} \\ &= \delta_{h-h(\beta)+h(\alpha),0}U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\alpha|}U^{h-h(\beta)} \\ &= \delta_{h-h(\beta)+h(\alpha),0}S_\alpha S_\beta^* U^h. \end{aligned}$$

□

In order to make our proof work, we also need to take into account the conditional expectation Θ from $\mathcal{Q}_2$ onto $\mathcal{D}_2$ described in [LL12]. We recall that this is uniquely determined by $\Theta((S_2^*)^i U^{-l} f U^{l'} S_2^{i'}) := \delta_{i,i'}\delta_{l,l'}(S_2^*)^i U^{-l} f U^l S_2^i$, where f is in $\mathcal{F}_2$. Moreover, it is there shown to be faithful too.

Lemma 3.2.8. *If $|\alpha| = |\beta|$, then $\Theta[S_\alpha S_\beta^* U^h]$ is either 0 or $S_\alpha S_\beta^* U^h$. In particular, Θ and E coincide on monomials $S_\alpha S_\beta^* U^h$ with $|\alpha| = |\beta|$.*

Proof. By direct computation. Indeed, we have

$$\begin{aligned}\Theta[S_\alpha S_\beta^* U^h] &= \Theta(U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\alpha|}U^{h-h(\beta)}) \\ &= \delta_{h(\alpha),-h+h(\beta)}U^{h(\alpha)}(S_2)^{|\alpha|}(S_2^*)^{|\alpha|}U^{h-h(\beta)}\end{aligned}$$

□

Theorem 3.2.9. *The diagonal subalgebra $\mathcal{D}_2 \subset \mathcal{Q}_2$ is a maximal abelian subalgebra.*

Proof. As usual, all we have to do is make sure that the relative commutant $\mathcal{D}_2' \cap \mathcal{Q}_2 = \ell_\infty(\mathbb{Z}) \cap \mathcal{Q}_2$ reduces to $\mathcal{D}_2$. Let $x \in \ell_\infty(\mathbb{Z}) \cap \mathcal{Q}_2$, then there exists a sequence $\{x_k\}$ converging normwise to x with each of the x_k of the form $\sum c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h$. As above, $x = E(x) = \lim_k E(x_k)$. Thanks to the former lemmata, we can rewrite $E(x_k)$ as $d_k + f_k$, where $d_k \in \mathcal{D}_2$ and f_k are all diagonal finite-rank operators. Now, being $d_k = \Theta(x_k)$, we see that d_k must converge to some $d \in \mathcal{D}_2$. But then f_k converge normwise to a diagonal compact operator, say k, which means $k = x - d$ is in $\mathcal{Q}_2$, hence $k = 0$, being $\mathcal{K}(\mathcal{H}) \cap \mathcal{Q}_2 = \{0\}$, and $x = d \in \mathcal{D}_2$. □

We now show that there is no conditional expectation from $\mathcal{Q}_2$ onto $\mathcal{O}_2$.

Theorem 3.2.10. *There is no unital conditional expectation from $\mathcal{Q}_2$ onto $\mathcal{O}_2$.*

Proof. Suppose that such a conditional expectation does exist. We want to show that this leads to $E(U)$ being U, which is obviously absurd. We shall work in any representation in which S_1 and S_2 are both pure, for instance the one described in [CL12]. If we compute E on the operator $US_1^n S_2 S_2^*(S_1^*)^n$ by using the commutation rule $S_2^n U = US_1^n$ we easily get to the equality

$$E[U]S_1^n S_2 S_2^*(S_1^*)^n = S_2^n S_1 S_2^*(S_1^*)^n$$

But on the other hand we also have that $US_1^n S_2 S_2^*(S_1^*)^n = S_2^n S_1 S_2^*(S_1^*)^n$. Accordingly, $E(U)$ and U must coincide on the direct sum of the subspaces $M_n \doteq S_1^n S_2 S_2^*(S_1^*)^n \mathcal{H}_\pi$, which can shown to be the whole $\mathcal{H}_\pi$ (see the proof of Proposition 3.2.17). □

3.2.2 The relative commutant of the generating isometry

This section is entirely devoted to proving that $C^*(S_2)' \cap \mathcal{Q}_2$ is trivial. We first observe that this is the same as proving that $C^*(S_1)' \cap \mathcal{Q}_2$ is trivial, merely because $\mathrm{ad}(U^*)(C^*(S_2)) = C^*(S_1)$. For this, we still need some preliminary definitions and results.

Given any $k \in \mathbb{N}$, we set $\mathcal{B}_2^k := \operatorname{span}\{S_\alpha S_\beta^* U^h \,||\alpha| = |\beta| = k, h \in \mathbb{Z}\}$.

Lemma 3.2.11. *Let $x \in \mathcal{B}_2^k$, then*

1. $(S_1^*)^k x S_1^k \in C^*(U)$;
2. *the sequence* $\{(S_1^*)^m x S_1^m\}$ *stabilizes to a scalar* $c_x \in \mathbb{C}$.

Proof. Without loss of generality, suppose that $x = S_\alpha S_\beta^* U^h$. We have that

$$(S_1^*)^k x S_1^k = \delta_{\alpha,\underline{1}} S_\beta^* U^h S_1^k.$$

If $h > 0$, then by using the relation $S_2 U = U S_1$, we see that the r.h.s. of the last expression is given by

$$\delta_{\alpha,\underline{1}} S_\beta^* S_\gamma U^{l(h)} = \delta_{\alpha,\underline{1}} \delta_{\beta,\gamma} U^{l(h)}$$

where γ is a multi-index of length k. If $h = 0$, we obtain

$$\delta_{\alpha,\underline{1}} S_\beta^* S_1^k = \delta_{\alpha,\underline{1}} \delta_{\beta,\underline{1}}.$$

When $h < 0$, by using the relations $U^* S_1 = S_2$ and $U^* S_2 = S_1 U^*$ we obtain

$$\delta_{\alpha,\underline{1}} S_\beta^* U^h S_1^k = \delta_{\alpha,\underline{1}} U^{l(h)} S_\gamma^* S_{\underline{1}} = \delta_{\alpha,\underline{1}} \delta_{\gamma,\underline{1}} U^{l(h)}$$

where γ is a multi-index of length k. We observe that in these cases we always have $|l(h)| \le |h|$, and this fact will be important in the sequel. For the second part of the thesis, we may suppose that $m > k + |h| + 1$. The needed computations can be made faster in the canonical representation (for brevity we write l instead of $l(h)$):

$$(S_1^*)^m U^l S_1^m e_j = (S_1^*)^m U^l e_{2^m j + 2^m - 1} = (S_1^*)^m e_{2^m j + 2^m - 1 + l}.$$

The expression above is non-zero if and only if $2^m j + 2^m - 1 + l = 2^m i + 2^m - 1$ for some i, that is to say $l = 2^m(i - j)$. But $m > k + h + 1$ and $|l| \le |h|$, therefore we finally get $i = j$ and $l = 0$.

□

Proposition 3.2.12. *Let $x \in \mathcal{Q}_2^{\mathbb{T}} = \overline{\operatorname{span}}\{S_\alpha S_\beta^* U^h \,||\alpha| = |\beta|, h \in \mathbb{Z}\}$. Then*

$$\lim_h (S_1^*)^h x S_1^h \in \mathbb{C}$$

Proof. By hypothesis there exists a sequence $x_k \in \mathcal{B}_2^{f(k)}$ that tends normwise to x. Choose a pair of natural numbers i and j. For any $k \in \mathbb{N}$ sufficiently larger than $f(i)$ and $f(j)$, by the former lemma we have that $(S_1^*)^k x_i S_1^k =: c_i, (S_1^*)^k x_j S_1^k =: c_j \in \mathbb{C}$. The sequence c_i is convergent since

$$|c_i - c_j| = \|(S_1^*)^k x_i S_1^k - (S_1^*)^k x_j S_1^k\| \le \|x_i - x_j\|.$$

We denote by c the limit. Now the sequence $(S_1^*)^h x S_1^h$ tends to c as well. □

For any non-negative integer i we now define the linear maps $F_i : \mathcal{Q}_2 \to \mathcal{Q}_2^{\mathbb{T}}$ given by

$$F_i(x) := \int_{\mathbb{T}} \tilde{\alpha}_z[x(S_1^*)^i]dz$$
$$F_{-i}(x) := \int_{\mathbb{T}} \tilde{\alpha}_z[S_1^i x]dz.$$

We observe that

$$F_i(x) = F_i(x)S_1^i(S_1^*)^i \tag{3.2.1}$$
$$F_{-i}(x) = S_1^i(S_1^*)^i F_{-i}(x). \tag{3.2.2}$$

Before proving the main result of the section, we also need to recall the following lemma, whose proof can be adapted verbatim from the original [Cun77, Proposition 1.10], where it is proved for the Cuntz algebras instead.

Proposition 3.2.13. *Let $x \in \mathcal{Q}_2$ be such that $F_i(x) = 0$ for all $i \in \mathbb{Z}$. Then $x = 0$.*

Now we have all the tools for completing our proof.

Theorem 3.2.14. *Let $w \in \mathcal{U}(\mathcal{Q}_2)$ such that $wS_1w^* = \lambda S_1$ for some $\lambda \in \mathbb{T}$. Then $w \in \mathbb{T}1$.*

Proof. First of all we observe that we also have $wS_1^*w^* = \overline{\lambda}S_1^*$, which in turn implies $wS_1^* = \overline{\lambda}S_1^*w$. We have that $S_1^*F_i(w)S_1 = \lambda F_i(w)$. Indeed, for $i \geq 0$,

$$\begin{aligned} S_1^*F_i(w)S_1 &= S_1^*\left(\int_{\mathbb{T}} \tilde{\alpha}_z[w(S_1^*)^i]dz\right)S_1 = \int_{\mathbb{T}} S_1^*\tilde{\alpha}_z[w(S_1^*)^i]S_1dz \\ &= \int_{\mathbb{T}} \tilde{\alpha}_z[S_1^*w(S_1^*)^iS_1]dz = \lambda\int_{\mathbb{T}} \tilde{\alpha}_z[w(S_1^*)^i]dz = \lambda F_i(w) \\ S_1^*F_{-i}(w)S_1 &= S_1^*\left(\int_{\mathbb{T}} \tilde{\alpha}_z[S_1^iw]dz\right)S_1 = \int_{\mathbb{T}} S_1^*\tilde{\alpha}_z[S_1^iw]S_1dz \\ &= \int_{\mathbb{T}} \tilde{\alpha}_z[S_1^*S_1^iwS_1]dz = \lambda\int_{\mathbb{T}} \tilde{\alpha}_z[S_1^iw]dz = \lambda F_{-i}(w) \end{aligned}$$

By Proposition 3.2.12 we obtain that for each $i \in \mathbb{Z}$ one has

$$\lim(S_1^*)^kF_i(w)S_1^k = \lim_k \lambda^kF_i(w) = (\lim_k \lambda^k)F_i(w) \in \mathbb{C}.$$

Equation (3.2.1)-(3.2.2) together imply that for $i \neq 0$ we have $F_i(w) = 0$. Now Proposition 3.2.13 applied to $w - F_0(w)$ gives the claim. □

We finally got the result we were looking for.

Theorem 3.2.15. *The relative commutant of $C^*(S_1)$ in $\mathcal{Q}_2$ is trivial, namely $C^*(S_1)\cap \mathcal{Q}_2 = \mathbb{C}1$. In particular, we also have that $\mathcal{O}_2' \cap \mathcal{Q}_2 = \mathbb{C}1$.*

Proof. The first part of the Theorem follows by taking $\lambda = 1$. For the second part of the statement this is actually an easy consequence of Theorem 3.2.15 and the fact $\mathcal{O}_2' \subset C^*(S_1)'$ (which in turn follows from the obvious inclusion $C^*(S_1) \subset \mathcal{O}_2$). □

Exactly as for the Cuntz algebras, the former Theorem yield the following result concerning outer automorphisms of $\mathcal{Q}_2$.

Proposition 3.2.16. *Let $\phi \in \mathrm{Aut}(\mathcal{Q}_2)$ be such that $\phi(S_i) = zS_i$ for some $z \in \mathbb{C}$. Then ϕ is an outer automorphism.*

In order to take a step further towards the study of $\mathcal{Q}_2$, especially as far as the properties of the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$ are concerned, it is worthwhile to recall a useful result proved by Larsen and Li in their aforementioned paper [LL12]. It says that a representation ρ of $\mathcal{O}_2$ extends to a representation of $\mathcal{Q}_2$ if and only if the unitary part of the Wold decomposition of $\rho(S_1)$ and $\rho(S_2)$ are unitarily equivalent. Accordingly, once the unitary parts of the Wold decompositions have been proved to be unitarily equivalent, then the isometries are unitarily equivalent too because of the relation $US_1 = S_2U$. As remarked by the authors themselves, this allows us to think of $\mathcal{Q}_2$ as a sort of symmetrized version of $\mathcal{O}_2$. Notably, the result applies to those representations π of $\mathcal{O}_2$ in which both $\pi(S_1)$ and $\pi(S_2)$ are pure. Moreover, in such cases the sought extension is unique. This is the content of the next proposition (cf. [LL12, Remark 4.2]).

Proposition 3.2.17. *Let π be a representation of $\mathcal{O}_2$ on the Hilbert space $\mathcal{H}_\pi$ such that $\pi(S_1)$ and $\pi(S_2)$ are both pure. Then there exists a unique $\widetilde{U} \in \mathcal{B}(\mathcal{H}_\pi)$ such that $\pi(S_2)\widetilde{U} = \widetilde{U}^2\pi(S_2)$.*

Proof. Thanks to the theorem recalled above, all we need to do is prove the uniqueness part in the statement. To this aim, we note that the following relations $\widetilde{U}S_1^nS_2S_2^*(S_1^*)^n = S_2^nS_1S_2^*(S_1^*)^n$ allow one to determine $\tilde{U}$ on the direct sum of the subspaces $M_n := S_1^nS_2S_2^*(S_1^*)^n\mathcal{H}_\pi$, which we now show to be the whole of $\mathcal{H}_\pi$. As S_1 is pure, we have $\{0\} = \bigcap_{n=0} S_1^{n+1}(S_1^*)^{n+1}\mathcal{H}_\pi$. As a consequence, we find that $P_{\bigcap_n M_n} = 0$, where $M_n := S_1^n(S_1^*)^n\mathcal{H}_\pi$. This in turn implies that $S_1^{n+1}(S_1^*)^{n+1}$ strongly converges to 0. But now we can rewrite $S_1^{n+1}(S_1^*)^{n+1}$ as follows

$$S_1^{n+1}(S_1^*)^{n+1} = 1 - (S_2S_2^* + S_1S_2S_2^*S_1^* + \ldots + S_1^nS_2S_2^*(S_1^*)^n)$$

which says that $(S_2S_2^* + S_1S_2S_2^*S_1^* + \ldots + S_1^nS_2S_2^*(S_1^*)^n)$ tends to 1 strongly, i.e. $\oplus_n M_n = \mathcal{H}_\pi$. □

One of the main results proved by Larsen and Li in [LL12] says that a representation ρ of $\mathcal{O}_2$ extends to a representation of $\mathcal{Q}_2$ if and only if the unitary part of the Wold decomposition of $\rho(S_1)$ and $\rho(S_2)$ are unitarily equivalent. Accordingly, once the unitary parts of the Wold decompositions have been proved to be unitarily equivalent, then the isometries are unitarily equivalent too because of the relation $US_1 = S_2U$.

We mention that the above Proposition could be obtained from the former result (see [ACR16, Section 3.2])).

Corollary 3.2.18. *If $x \in \mathcal{Q}_2$ is such that $x = \tilde{\varphi}(x)$, then x is a scalar.*

Proof. By hypothesis, we have $x = S_1xS_1^* + S_2xS_2^*$. If we now multiply the former equality by S_i on the right, we easily get $xS_i = S_ix$, with $i = 1, 2$. Hence, x must be trivial, being an element of the relative commutant $\mathcal{O}_2{}' \cap \mathcal{Q}_2$. □

As a matter of fact, much more can be said of the canonical shift. In fact, it turns out that it enjoys the so-called shift property, i.e. $\bigcap_k \tilde{\varphi}^k(\mathcal{Q}_2) = \mathbb{C}1$, whence its name. This important property should have first been singled out by R. T. Powers, who called it strong ergodicity, but we do not have a precise reference for the reader. The canonical shifts of $\mathcal{O}_n$ are of course known to be strongly ergodic, see [Lac93] for a full coverage of the topic. The proof we are about to give, though, does not rely on this last statement, which means our result can be thought of as an actual generalization at least as far as $\mathcal{O}_2$ is concerned.

Theorem 3.2.19. *The canonical endomorphism $\tilde{\varphi}$ of $\mathcal{Q}_2$ is a shift, i.e. $\bigcap_k \tilde{\varphi}^k(\mathcal{Q}_2) = \mathbb{C}1$.*

Proof. Since $\tilde{\varphi}^k(x) = \sum_{\gamma:|\gamma|=k} S_\gamma x S_\gamma^*$, the equality $\tilde{\varphi}^k(x)S_\alpha S_\beta^* = S_\alpha S_\beta^* \tilde{\varphi}^k(x)$ is straightforwardly checked to hold true for every pair of multi-indices α and β with $|\alpha| = |\beta| = k$. This says that $\tilde{\varphi}^k(\mathcal{Q}_2)$ is contained in $(\mathcal{F}_2^k)' \cap \mathcal{Q}_2$ for every k. But then we have the chain of inclusions $\bigcap_k \tilde{\varphi}^k(\mathcal{Q}_2) \subset \bigcap_k (\mathcal{F}_2^k)' \cap \mathcal{Q}_2 \subset \mathcal{F}_2' \cap \mathcal{Q}_2 \subset \mathcal{D}_2' \cap \mathcal{Q}_2 = \mathcal{D}_2$, where the last equality depends upon $\mathcal{D}_2$ being a maximal abelian subalgebra of $\mathcal{Q}_2$. Therefore $\mathcal{F}_2' \cap \mathcal{Q}_2 = \mathcal{F}_2' \cap \mathcal{D}_2 \subset \mathcal{F}_2' \cap \mathcal{F}_2 = Z(\mathcal{F}_2) = \mathbb{C}1$. □

In particular, in the proof above we saw the equality $\mathcal{F}_2' \cap \mathcal{Q}_2 = \mathbb{C}1$, i.e. $\mathcal{F}_2$ is an irreducible subalgebra of $\mathcal{Q}_2$, a fact worth stressing.

Remark 3.2.20. The former theorem says in particular that $\tilde{\varphi}$ is not surjective in a rather strong sense. We can be more precise by observing that U is not in $\tilde{\varphi}(\mathcal{Q}_2)$. Indeed, by maximality of $C^*(U)$, any inverse image of U should lie in $C^*(U)$, but the restriction of $\tilde{\varphi}$ to $C^*(U)$ does not yield a homeomorphism of $\mathbb{T}$.

As a corollary, we can also state yet another result of irreducibility, for the Cuntz algebra $\mathcal{O}_2$ is by no means the sole remarkable subalgebra of $\mathcal{Q}_2$ we know of to have a trivial relative commutant.

Proposition 3.2.21. *The fixed-point subalgebra $\mathcal{Q}_2^{\mathbb{T}}$ is irreducible in $\mathcal{Q}_2$, i.e. $(\mathcal{Q}_2^{\mathbb{T}})' \cap \mathcal{Q}_2 = \mathbb{C}1$.*

Proof. A straightforward consequence of the inclusion $\mathcal{F}_2 \subset \mathcal{Q}_2^{\mathbb{T}}$. □

3.3 Extending endomorphisms of the Cuntz algebra

3.3.1 Uniqueness of the extensions

In this section we discuss the problem of extending endomorphisms of $\mathcal{O}_2$ to endomorphisms of $\mathcal{Q}_2$. More precisely, we give necessary and sufficient conditions for an extension to exist. Let V is a unitary of $\mathcal{O}_2$ such that the associated endomorphism λ_V extends to an endomorphism $\tilde{\lambda}$ of $\mathcal{Q}_2$, then one must have $VUS_2 = VS_1 = \lambda_V(S_1) = \tilde{\lambda}(S_1) = \tilde{\lambda}(U)\lambda_V(S_2) = \tilde{\lambda}(U)VS_2$. Therefore, $\tilde{\lambda}(U)VS_2 = VUS_2$ and thus $U^*V^*\tilde{\lambda}(U)VS_2 = S_2$. Thus, setting $W = U^*V^*\tilde{\lambda}(U)V$, it holds $WS_2 = S_2$ and $\tilde{\lambda}(U) = VUWV^*$. We now examine whether such extensions exist. We always have

$$VS_2S_2^*V^* + VUWV^*VS_2S_2^*V^*VW^*U^*V^* = V(S_2S_2^* + US_2S_2^*U^*)V^* = VV^* = 1$$

and we must have

$$VS_2\tilde{\lambda}(U) = \tilde{\lambda}(U^2)VS_2$$

or, equivalently,

$$VS_2VUWV^* = (VUWV^*)^2VS_2 = VUWV^*VUWV^*VS_2 = VUWUWS_2 = VUWUS_2\,.$$

We have thus shown the following result.

Proposition 3.3.1. *Let $V \in \mathcal{U}(\mathcal{O}_2)$ and let $\lambda_V \in \mathrm{End}(\mathcal{O}_2)$ be the associated endomorphism. Then λ_V extends to an endomorphism of $\mathcal{Q}_2$ if and only if there exists a unitary $W \in \mathcal{Q}_2$ such that $WS_2 = S_2$ and $S_2VUWV^* = UWUS_2$. For any such W, we have an extension $\tilde{\lambda} = \tilde{\lambda}_{V,W}$ with $\tilde{\lambda}(U) = VUWV^*$.*

We are going to see that the unitary W defined above is uniquely determined in all the cases we have examined. Furthermore, the endomorphism $\tilde{\lambda}$ is necessarily injective because of $\mathcal{Q}_2$ being simple. Moreover, if λ_V is an automorphism of $\mathcal{O}_2$, then $\tilde{\lambda}$ is surjective if and only if the associated W is contained in $\tilde{\lambda}(\mathcal{Q}_2)$. Moreover, for the extensions built above the following composition rule holds:

$$\tilde{\lambda}_{V,W} \circ \tilde{\lambda}_{V',W'} = \tilde{\lambda}_{\lambda_V(V')V, WV^*\tilde{\lambda}_{V,W}(W')V}$$

As an example, we have

$$\tilde{\varphi} = \lambda_{\theta, U^*\theta U^2\theta}$$

where $\theta = \sum_{i,j=1}^2 S_iS_jS_i^*S_j^* \in \mathcal{U}(\mathcal{F}_2^2)$ is the self-adjoint unitary flip.

It is interesting to note that the extensions of the gauge automorphisms we have considered all work with $W = 1$. This is not a case. In fact, the converse also holds true.

Proposition 3.3.2. *Let $V \in \mathcal{U}(\mathcal{O}_2)$. If the associated endomorphism $\lambda_V \in \mathrm{End}(\mathcal{O}_2)$ extends to $\tilde{\lambda}_{V,1}$, that is to say the choice $W = 1$ does yield an extension, then $V = z1$, for some $z \in \mathbb{T}$.*

Proof. If we put $W = 1$ in the equality $S_2VUWV^* = UWUS_2$, we get $S_2VUV^* = U^2S_2$. But $U^2S_2 = S_2U$, and so we must have $S_2VUV^* = S_2U$. Hence $VUV^* = U$, i.e. V commutes with U. Since V is a unitary, we also have $V \in C^*(U)' \cap \mathcal{O}_2$, which concludes the proof. ∎

Extensions of the identity map of $\mathcal{O}_2$, which obviously correspond to $V = 1$, may be looked at more closely. If we define $W := U^*\tilde{\lambda}(U)$, we find that W is a unitary in $\mathcal{Q}_2$ such that $\tilde{\lambda}(U) = UW$, $WS_2 = S_2$ and $WS_1 = S_1W$. Indeed, $S_2S_2^* + UWS_2S_2^*W^*U^* = S_2S_2^* + US_2S_2^*U^* = 1$ and $S_2UW = (UW)^2S_2$, so that $U^2S_2W = UWUWS_2$ and thus $US_2W = WUS_2$. Hence, $S_1W = WS_1$, as stated. Obviously, the trivial choice $W = 1$ corresponds to the trivial extension.

Proposition 3.3.3. *If $W \in \mathcal{U}(\mathcal{Q}_2)$ is such that $WS_2 = S_2$ and $WS_1 = S_1W$, then $W = 1$.*

Proof. This is in fact a straightforward application of the fact that $C^*(S_1)' \cap \mathcal{Q}_2 = \mathbb{C}1$. □

We are at last in a position to prove the following result that says that a non-trivial endomorphism of $\mathcal{Q}_2$ cannot fix $\mathcal{O}_2$ pointwise.

Theorem 3.3.4. *If $\Lambda \in \mathrm{End}(\mathcal{Q}_2)$ is such that $\Lambda \restriction_{\mathcal{O}_2} = \mathrm{id}_{\mathcal{O}_2}$, then $\Lambda = \mathrm{id}_{\mathcal{Q}_2}$.*

Proof. A straightforward application of the former proposition. □

As a simple corollary, we can also get the following property of the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$.

Corollary 3.3.5. *If $\Lambda_1 \in \mathrm{Aut}(\mathcal{Q}_2)$ and $\Lambda_2 \in \mathrm{End}(\mathcal{Q}_2)$ are such that $\Lambda_1 \restriction_{\mathcal{O}_2} = \Lambda_2 \restriction_{\mathcal{O}_2}$, then $\Lambda_1 = \Lambda_2$. In particular, Λ_2 is an automorphism as well.*

Proof. Just apply the above theorem to the endomorphism $\Lambda_1^{-1} \circ \Lambda_2$, which restricts trivially to $\mathcal{O}_2$. □

In particular, the extensions of both the flip-flop and the gauge automorphisms are unique.

Of course there are automorphisms of $\mathcal{Q}_2$ that do not leave $\mathcal{O}_2$ globally invariant. The most elementary example we can come up with is probably $\mathrm{Ad}(U)$. Indeed, $\mathrm{Ad}(U)S_1 = US_1U^* = S_2 = US_1$, $\mathrm{Ad}(U)S_2 = US_2U^* = S_1U^* = U^*S_2$ Hence, $\mathrm{ad}(U)(\mathcal{O}_2)$ is not contained in $\mathcal{O}_2$, because S_1U^* is not in $\mathcal{O}_2$. Even more can be said. Indeed, $\mathrm{Ad}(U)(\mathcal{F}_2)$ is not contained in $\mathcal{O}_2$ either. This is seen as easily as before, since for instance $\mathrm{Ad}(U)(S_1S_2^*) = US_1S_2^*U^*$ does not belong to O_2 although $S_1S_2^*$ belongs to $\mathcal{F}_2$. Given that $US_1S_2^*U^* = S_2US_2^*U^* = S_2US_1^*UU^* = S_2US_1^*$, if $US_1S_2^*U^*$ were in $\mathcal{O}_2$, then $U = S_2^*S_2US_1^*S_1$ would in turn be in $\mathcal{O}_2$, which is not. Even so, $\mathrm{Ad}(U)$ does leave the diagonal subalgebra $\mathcal{D}_2$ globally invariant. This can be shown by means of easy computations involving the projections of $\mathcal{D}_2^k := \mathrm{span}\{S_\alpha S_\alpha^* \text{ s.t. } \ell(\alpha) = k\}$ for every $k \in \mathbb{N}$.

We would like to end this section by remarking that for each $\Lambda \in \mathrm{End}(\mathcal{Q}_2)$ there still exists a unique $u_\Lambda \in \mathcal{U}(\mathcal{Q}_2)$ such that $\Lambda(S_2) = u_\Lambda S_2$ and $\Lambda(S_1) = u_\Lambda S_1$, which is simply given by $u_\Lambda = \Lambda(S_1)S_1^* + \Lambda(S_2)S_2^*$. Furthermore, Λ leaves $\mathcal{O}_2$ globally invariant if and only if $u_\Lambda \in \mathcal{O}_2$. This allows us to regard the map $\mathrm{End}(\mathcal{Q}_2) \ni \Lambda \to u_\Lambda \in \mathcal{U}(\mathcal{Q}_2)$

as a generalization of the well-known Cuntz-Takesaki correspondence. Nevertheless, this map is decidedly less useful for $\mathcal{Q}_2$ than it is for $\mathcal{O}_2$, not least because it is not surjective. In other words, there exist unitaries u in $\mathcal{U}(\mathcal{Q}_2)$ such that the correspondence $S_1 \to uS_1$, $S_2 \to uS_2$ do not extend to any endomorphism of $\mathcal{Q}_2$. Examples of such u are even found in $\mathcal{U}(\mathcal{O}_2)$, as we are going to see in the next section, where we shall give a complete description of the extensible Bogoljubov automorphisms. For the time being we observe that if a unitary $u \in \mathcal{U}(\mathcal{Q}_2)$ does give rise to an endomorphism Λ_u, the equation $uUS_2 = \Lambda_u(U)uS_2$ must be satisfied. This says that $\Lambda_u(U) = uUWu^*$ for some $W \in \mathcal{U}(\mathcal{Q}_2)$ such that $WS_2 = S_2$ and $S_2uUWu^* = UWUS_2$. By the same computations as at the beginning of the section, the converse is also seen to be true. Hence we obtain a complete if hitherto unmanageable description of $\mathrm{End}(\mathcal{Q}_2)$. At any rate, our guess is that the above equations are hardly ever verified unless u is of a very special form, such as $u = v\tilde{\varphi}(v^*)$ for any $v \in \mathcal{U}(\mathcal{Q}_2)$, corresponding to inner automorphisms, $u = e^{i\theta}1$, corresponding to the gauge automorphisms, or $u = S_2S_2^*U^* + US_2S_2^*$, corresponding to the flip-flop. In fact, this prediction is partly supported by the result in the negative obtained in the next section. Moreover, it is still not clear at all whether the map $\mathrm{End}(\mathcal{Q}_2) \ni \Lambda \to u_\Lambda$ is injective, although its restriction to $\mathrm{Aut}(\mathcal{Q}_2)$ certainly is.

3.3.2 Extensible Bogoljubov automorphisms

At the beginning of this chapter we gave some example of endomorphisms of $\mathcal{O}_2$ that extend to $\mathcal{Q}_2$. Now we are going to study a particular family of automorphisms of $\mathcal{O}_2$, namely the Bogoljubov automorphisms (see Section 1.5), and precisely determine which ones extends to automorphisms of $\mathcal{Q}_2$. We begin with exhibiting some automorphisms that do not extends. If we denote by $\sigma_{\alpha,\beta}$ the automorphism of Ω_2 defined by $\sigma_{\alpha,\beta}(S_1) = \alpha S_1$ and $\sigma_{\alpha,\beta}(S_2) = \beta S_2$ for any given $\alpha, \beta \in \mathbb{T}$, we have the following proposition.

Proposition 3.3.6. *The automorphisms $\sigma_{\alpha,\beta} \in \mathrm{Aut}(\Omega_2)$ defined above extend to endomorphisms of $\mathcal{Q}_2$ if and only if $\alpha = \beta$.*

Proof. Since S_1 and S_2 are unitarily equivalent in $\mathcal{Q}_2$, their images αS_1 and βS_2 would be unitarily equivalent as well if an extension of $\sigma_{\alpha,\beta}$ existed. In particular, we would find $\{\alpha\} = \sigma_p(\alpha S_1) = \sigma_p(\beta S_2) = \{\beta\}$, which is absurd unless $\alpha = \beta$, in which case the corresponding endomorphism does extend being but a gauge automorphism. □

It is no surprise that the same proof as above covers the case of the so-called anti-diagonal automorphisms. These are simply given by $\rho_{\alpha,\beta}(S_1) = \alpha S_2$ and $\rho_{\alpha,\beta}(S_2) = \beta S_1$ for any given $\alpha, \beta \in \mathbb{T}$. Again, an automorphism $\rho_{\alpha,\beta}$ extends precisely when $\alpha = \beta$. To complete the picture, we shall presently determine which Bogoljubov automorphisms of Ω_2 extend to endomorphisms of $\mathcal{Q}_2$. A suitable adaptation of some of the techniques developed by Matsumoto and Tomiyama in [MT93] will be again among the ingredients to concoct the proof of the main result of this section. This says that the extensible Bogoljubov automorphisms are precisely the flip-flop, the gauge automorphisms, and their products, which altogether form a group isomorphic with the direct product $\mathbb{T}\times\mathbb{Z}_2$.

To this aim, consider a unitary matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(\mathbb{C}^2).$$

and let $\alpha = \lambda_A$ be the corresponding automorphism of Ω_2, i.e. $\alpha(S_1) = aS_1 + cS_2 = (aU + c)S_2$ and $\alpha(S_2) = bS_1 + dS_2 = (bU + d)S_2$. Set $f(U) = (bU + d)$ for short. The condition $S_2U = U^2S_2$ implies that $f(U)S_2\alpha(U) = \alpha(U)^2 f(U)S_2$. Suppose that α is extensible and denote by $\tilde{\alpha}$ such an extension. Finally, set $\tilde{U} \doteq \tilde{\alpha}(U)$, $\tilde{S}_2 \doteq \alpha(S_2)$, $\tilde{S}_1 \doteq \alpha(S_1)$. From now on we shall always be focusing on the case where a, b, c, d are all different from zero. That said, the first thing we need to prove is the extension is unique provided that it exists.

Lemma 3.3.7. *If λ_A extends, then its extension is unique.*

Proof. By Proposition 3.2.17 all we need to check is that $\tilde{S}_1$ is pure as an isometry acting on $\ell_2(\mathbb{Z})$. This entails ascertaining that $\bigcap_n \mathrm{Ran}[\tilde{S}_1^n(\tilde{S}_1^*)^n] = \{0\}$. To this aim, let us set $M_n \doteq \mathrm{Ran}[\tilde{S}_1^n(\tilde{S}_1^*)^n]$. As $M_{n+1} \subset M_n$, we have that $E_{\bigcap_n M_n} = \lim E_{M_n}$ strongly. Thus we are led to show $\lim_n E_{M_n} = 0$. For this it is enough to prove $\lim_n \|E_{M_n}e_k\| = 0$ for every k. Now the powers of $\tilde{S}_1$ are given by $\tilde{S}_1^n = \sum_{|\alpha|=n} c_\alpha S_\alpha$, where $c_\alpha \doteq a^{n_1(\alpha)}c^{n_2(\alpha)} \in \mathbb{C}^*$ with $n_1(\alpha)$ being the number of 1's occurring in α and $n_2(\alpha)$ the number of 2's occurring in α. We set $L \doteq \max\{|a|, |c|\}$ and observe that $L < 1$ by the hypotheses on the unitary matrix A. We have that

$$\|(\tilde{S}_1^*)^n e_k\| = |c_{\alpha(k)}| \leq L^n \to 0 \quad n \to +\infty$$

for a unique coefficient $c_{\alpha(k)}$ that depends on k (this is actually a consequence of the fact that $\sum_{|\alpha|=k} S_\alpha S_\alpha^* = 1$). This in turn implies the claim. □

In light of the previous result, it is a very minor abuse of notation to denote by λ_A also its extension to $\mathcal{Q}_2$ when existing.

Lemma 3.3.8. *If λ_A extends, then $\tilde{U} \in \mathcal{Q}_2^{\mathbb{T}}$.*

Proof. Suppose that $\tilde{U}$ is not in $\mathcal{Q}_2^{\mathbb{T}}$. Then by definition there must exist a non-trivial gauge automorphism $\widetilde{\alpha_\theta}$ such that $\widetilde{\alpha_\theta}(\tilde{U}) \neq \tilde{U}$. By applying $\widetilde{\alpha_\theta}$ to both sides of the equalities $\tilde{S}_2\tilde{U} = \tilde{U}^2\tilde{S}_2$ and $\tilde{S}_2\tilde{S}_2^* + \tilde{U}\tilde{S}_2\tilde{S}_2^*\tilde{U}^* = 1$, we also get

$$\widetilde{\alpha_\theta}(\tilde{U})^2\tilde{S}_2 = \tilde{S}_2\widetilde{\alpha_\theta}(\tilde{U})$$
$$\tilde{S}_2\tilde{S}_2^* + \widetilde{\alpha_\theta}(\tilde{U})\tilde{S}_2\tilde{S}_2^*\widetilde{\alpha_\theta}(\tilde{U})^* = 1$$

which together say there exists an endomorphism $\Lambda \in \mathrm{End}(\mathcal{Q}_2)$ such that $\Lambda(S_2) = \tilde{S}_2$ and $\Lambda(U) = \widetilde{\alpha_\theta}(\tilde{U})$. Now $\Lambda(S_1) = \Lambda(US_2) = \Lambda(U)\Lambda(S_2) = \widetilde{\alpha_\theta}(\tilde{U})\tilde{S}_2 = e^{-i\theta}\widetilde{\alpha_\theta}(\tilde{U}\tilde{S}_2) = e^{-i\theta}\widetilde{\alpha_\theta}(\tilde{S}_1) = \tilde{S}_1 = \lambda_A(S_1)$. A contradiction is thus arrived at because Λ and $\widetilde{\lambda_A}$ are different maps. □

Now we introduce some lemmas to prove that $\tilde{\alpha}(U)$ is actually contained in $C^*(U)$.

Lemma 3.3.9. *For any $x \in \mathcal{B}_2^k$, the element $(\tilde{S}_2^*)^k x \tilde{S}_1^k$ belongs to $C^*(U)$.*

Proof. Suppose that $x = S_\alpha S_\beta^* U^h$, where $|\alpha| = |\beta| = k$. If $h \geq 0$, we have that $(\tilde{S}_2^*)^k x \tilde{S}_1^k = (\tilde{S}_2^*)^k S_\alpha S_\beta^* U^h \tilde{S}_1^k$ is a polynomial in U. The case $h \leq 0$ can be handled with similar computations. □

Lemma 3.3.10. *Let $x \in \mathcal{Q}_2^{\mathbb{T}}$ such that the sequence $(\tilde{S}_2^*)^k x \tilde{S}_1^k$ converges to an element z. Then $z \in C^*(U)$.*

Proof. Let $\{y_k\}_{k\geq 0}$ be a sequence such that $y_k \in \mathcal{B}_2^k$ and $y_k \to x$ normwise. Then the thesis follows from the following inequality

$$\|z - (\tilde{S}_2^*)^k y_k \tilde{S}_1^k\| \leq \|z - (\tilde{S}_2^*)^k x \tilde{S}_1^k\| + \|(\tilde{S}_2^*)^k (x - y_k) \tilde{S}_1^k\| .$$

□

Lemma 3.3.11. *We have that $\tilde{U} \in C^*(U)$.*

Proof. By applying $\widetilde{\lambda_A}$ to the identity $US_1^k = S_2^k U$ we get $\tilde{U}\tilde{S}_1^k = \tilde{S}_2^k \tilde{U}$. For all $k \in \mathbb{N}$ we have that $(\tilde{S}_2^*)^k \tilde{U} \tilde{S}_1^k = \tilde{U}$. Therefore, $\tilde{U}$ is in $C^*(U)$ thanks to Lemma 3.3.10, applied to $x = \tilde{U}$. □

We have verified that $\alpha(U) = g(U)$ for some $g \in C(\mathbb{T})$, which turns out to be vital in proving the following result.

Theorem 3.3.12. *If $\alpha \in \mathrm{Aut}(\Omega_2)$ is a Bogoljubov automorphism, then α extends to $\mathcal{Q}_2$ if and only if α is the flip-flop, a gauge automorphism, or a composition of these two.*

Proof. By the discussion at the beginning of this section it is enough to consider the case in which a, b, c, d are all different from zero. All the computations are henceforth made in the canonical representation. The condition $f(U)S_2\alpha(U) = \alpha(U)^2 f(U)S_2$ yields

$$f(U)S_2 g(U) = g(U)^2 f(U)S_2$$
$$f(U)g(U^2)S_2 = f(U)g(U)^2 S_2.$$

Since the point spectrum of U is empty, $f(U)$ is always injective, unless $b = d = 0$, in which case A is not unitary. Thus $g(U^2)S_2 = g(U)^2 S_2$. At the function level we must then have $g(z^2) = g(z)^2$ for every $z \in \mathbb{T}$. By continuity, we find that $g(z) = z^l$, for this see e.g. the Appendix. Therefore $g(U) = U^l$. We have that $\alpha(S_1) = aS_1 + cS_2 = bU^{l+1}S_2 + dU^l S_2$. If we compute the above equality on the vectors e_m, we get

$$ae_{2m+1} + ce_{2m} = be_{2m+l+1} + de_{2m+l}.$$

which is to be satisfied for each $m \in \mathbb{Z}$. Therefore, there are only two possibilities to fulfill these conditions:

1. $l = 1$, and $a = d \neq 0$, $b = c = 0$;
2. $l = -1$, and $b = c \neq 0$, $a = d = 0$.

The first corresponds to gauge automorphisms, whilst the second to the flip-flop and its compositions with gauge automorphisms.

□

3.4 Outer automorphisms

In this section the group $\mathrm{Out}(\mathcal{Q}_2)$ is shown to be non-trivial. More precisely, it turns out to be a non-abelian uncountable group. A thorough description of its structure, though, is still missing.

3.4.1 Gauge automorphisms and the flip-flop

Below the flip-flop and non-trivial gauge automorphisms are proved to be outer. In fact, this parallels analogue known results for $\mathcal{O}_2$. Since gauge automorphisms are more easily dealt with, we start our discussion focusing on them first.

Theorem 3.4.1. *The extensions to $\mathcal{Q}_2$ of the non-trivial gauge automorphisms of $\mathcal{O}_2$ are still outer automorphisms.*

Proof. This is a consequence of Proposition 3.2.16. □

Among other things, we also gain the additional information that $\mathrm{Out}(\mathcal{Q}_2)$ is an uncountable group, in that different gauge automorphisms give raise to distinct classes in $\mathrm{Out}(\mathcal{Q}_2)$. Indeed, if $\tilde{\alpha}_\theta$ and $\tilde{\alpha}_{\theta'}$ are two different gauge automorphisms, then there cannot exist any unitary $u \in \mathcal{Q}_2$ such that $u\tilde{\alpha}_\theta(x)u^* = \tilde{\alpha}_{\theta'}(x)$ for every $x \in \mathcal{Q}_2$. For such a u should commute with both U and S_2, and so it should be trivial, i.e. $\tilde{\alpha}_\theta = \tilde{\alpha}_{\theta'}$. For the sake of completeness we should also mention that every separable traceless C^*-algebra is actually known to have uncountable many outer automorphisms [Phi87].

Remark 3.4.2. Notably, the former result also provides a new and simpler proof of the well-known fact that the gauge automorphisms on $\mathcal{O}_2$ are outer. However, the case of a general $\mathcal{O}_n$ cannot be recovered from our discussion, and must needs be treated separately, as already done elsewhere.

As for the flip-flop, instead, we start our discussion by showing it is a weakly inner automorphism, which is the content of the next result.

Proposition 3.4.3. *The extension of the flip-flop to $\mathcal{Q}_2$ is a weakly inner automorphism.*

Proof. By definition, we only have to produce a representation π of $\mathcal{Q}_2$ such that $\tilde{\lambda}_f$ is implemented by a unitary in $\pi(\mathcal{Q}_2)''$. The canonical representation does this job well. For if $V \in U(\ell_2(\mathbb{Z}))$ is the self-adjoint unitary given by $Ve_k \doteq e_{-k-1}$, the equalities $VS_1V^* = S_2$ and $VS_2V^* = S_1$ are both easily checked. Since the canonical representation is irreducible, the proof is thus complete. □

This result should also be compared with a well-known theorem by Archbold [Arc79] that the flip-flop is weakly inner on $\mathcal{O}_2$.

Remark 3.4.4. The unitary V as defined above can be rewritten as $V = \mathcal{P}U = U^*\mathcal{P}$, where $\mathcal{P}$ is the self-adjoint unitary given by $\mathcal{P}e_k = e_{-k}$, $k \in \mathbb{Z}$. Obviously, V is in $\mathcal{Q}_2$ if and only if $\mathcal{P}$ is. We shall prove that $\mathcal{P}$ is not in $\mathcal{Q}_2$ in a while. At any rate, we observe the equality $V(S_1VS_1^* + S_2VS_2^*)^* = S_1S_2^* + S_2S_1^* \doteq f \in \mathcal{O}_2$, which is immediately checked, and $fV = S_1VS_1^* + S_2VS_2^* = Vf$. Finally, it is worth noting that $U = \lambda_{\mathbb{Z}}(1)$, and that $\mathcal{P}$ is the canonical intertwiner between $\lambda_{\mathbb{Z}}$ and $\rho_{\mathbb{Z}}$. In this picture, $\mathcal{Q}_2$ is thus the concrete C*-algebra on $\ell_2(\mathbb{Z})$ generated by $C_r^*(\mathbb{Z})$ and the copy of $\mathcal{O}_2$ provided by the canonical representation.

In spite of being weakly inner, $\tilde{\lambda}_f$ is an outer automorphism, as is its restriction to $\mathcal{O}_2$. To prove that, we first need to show that the unitary V above is up to multiplicative scalars the unique operator in $B(\ell_2(\mathbb{Z}))$ that implements the flip-flop.

Proposition 3.4.5. *If $W \in B(\ell_2(\mathbb{Z}))$ is such that* $\mathrm{ad}(W) \upharpoonright_{\mathcal{Q}_2} = \tilde{\lambda}_f$*, then $W = \lambda V$ for some $\lambda \in \mathbb{T}$.*

Proof. First note that we must have $\mathrm{ad}(W^2) = id_{B(\ell_2(\mathbb{Z}))}$ since the flip-flop is an involutive automorphism and $\mathcal{Q}_2'' = B(\ell_2(\mathbb{Z}))$. Hence W^2 is a multiple of 1, and therefore there is no loss of generality if we also assume that $W^2 = 1$, i.e. $W = W^*$. From the relation $WS_1W = S_2$, we get $S_1We_0 = We_0$. Hence $We_0 = \lambda e_{-1}$ for some $\lambda \in \mathbb{T}$. From $e_0 = W^2e_0 = \lambda We_{-1}$ we get $We_{-1} = \bar{\lambda}e_0$. We now show that either $\lambda = 1$ or $\lambda = -1$. Indeed, from $UW = WU^*$ it follows that $We_{-1} = WU^*e_0 = UWe_0 = U(\lambda e_{-1}) = \lambda e_0 = \bar{\lambda}e_0$, which in turn implies that λ is real. Of course, we only need to deal with the case $\lambda = 1$. The conclusion is now obtained at once if we use the equality $WU = U^*W$ inductively, for $We_{k+1} = WUe_k = U^*We_k = U^*e_{-k-1} = e_{-k-2}$, as maintained. □

Remark 3.4.6. Of course, the uniqueness of V could also have been obtained faster merely by irreducibility of $\mathcal{Q}_2$. However, the proof displayed above has the advantage of showing how we came across the operator V.

Here finally follows the theorem on the outerness of the flip-flop.

Theorem 3.4.7. *The extension of the flip-flop is an outer automorphism.*

Proof. Thanks to the former result, all we have to prove is that $\mathcal{P}$ is not in $\mathcal{Q}_2$, which entails checking that $\mathcal{P}$ cannot be a norm limit of a sequence x_k of operators of the form $x_k = \sum_k c_k S_{\alpha_k} S_{\beta_k^*} U^{h_k}$. Indeed, if this were the case, we should have $\varepsilon > \|\mathcal{P} - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h\|$ for some finite sum of the kind $\sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta U^h$, with $\varepsilon > 0$ as small as needed. If so, we would also find the inequality

$$\|e_{-n} - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_n\| = \|\mathcal{P}e_n - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_n\| < \varepsilon$$

This inequality, though, becomes absurd as soon as $\varepsilon < 1$ and n is sufficiently large, i.e. n is bigger than the largest value of $|h|$, for we would have $\|e_{-n} - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_n\|^2 = 1 + \|\sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_n\|^2 \geq 1$, as $\sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_n \in \mathcal{H}_+$. □

Now, as we know $\tilde{\lambda}_f$ is outer, we would also like to raise the question whether there exists a representation of $\mathcal{Q}_2$ in which $\tilde{\lambda}_f$ is not unitarily implemented. The answer would indeed complete our knowledge of $\tilde{\lambda}_f$ itself.

3.4.2 A general result

We saw above that the flip-flop is an outer automorphism. This is not an isolated case, for every automorphism that takes U to its adjoint must in fact be outer. This is the content of the next result.

Theorem 3.4.8. *Every automorphism $\alpha \in \mathrm{Aut}(\mathcal{Q}_2)$ such that $\alpha(U) = U^*$ is an outer automorphism.*

Proof. All we have to prove is that there is no unitary $W \in \mathcal{Q}_2$ such that $WUW^* = U^*$. To this aim, we will be working in the canonical representation. If $W \in B(\mathcal{H})$ is a unitary operator such that $WUW = U^*$, then we have $WUW^* = \mathcal{P}U\mathcal{P}$, hence $\mathcal{P}WU(\mathcal{P}W)^* = U$, which says that $\mathcal{P}W$ commutes with U. Therefore, $\mathcal{P}W = f(U)$ for some $f \in L^\infty(\mathbb{T})$ with $|f(z)| = 1$ a.e. with respect to the Haar measure of $\mathbb{T}$, hence $W = \mathcal{P}f(U)$. Then we need to show that such a W cannot be in $\mathcal{Q}_2$. If f is a continuous function, there is not much to say, for $\mathcal{P}f(U) \in \mathcal{Q}_2$ would immediately imply that $\mathcal{P} = \mathcal{P}f(U)f(U)^*$ is in $\mathcal{Q}_2$ as well, which we know is not the case. The general case of an essentially bounded function is dealt with in much the same way apart from some technicalities to be overcome. Given any $f \in L^\infty(\mathbb{T})$ and $\varepsilon > 0$, thanks to Lusin's theorem we find a closed set $C_\varepsilon \subset \mathbb{T}$ such that $\mu(\mathbb{T} \setminus C_\varepsilon) < \epsilon$ and $f \restriction_{C_\varepsilon}$ is continuous. This in turn guarantees that there exists a continuous function $g_\varepsilon \in C(\mathbb{T}, \mathbb{T})$ that coincides with f on C_ε by an easy application of the Tietze extension theorem. If $\mathcal{P}f(U)$ is in $\mathcal{Q}_2$, then $\mathcal{P}f(U)g_\varepsilon(U)^*$ is also in $\mathcal{Q}_2$. Note that $f\bar{g}_\varepsilon = 1 + h_\varepsilon$, where h_ε is a suitable function whose support is contained in $\mathbb{T} \setminus C_\varepsilon$. In particular, we can rewrite $\mathcal{P}f(U)g_\varepsilon(U)^*$ as $\mathcal{P} + \mathcal{P}h_\varepsilon(U)$. If the latter operator were in $\mathcal{Q}_2$, then we could find an operator of the form $\sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h$ such that $\|\mathcal{P} + \mathcal{P}h_\varepsilon(U) - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h\| < \varepsilon$. If N is any natural number greater then the maximum value of $|h|$ as h runs over the set the above summation is performed on, we should have $\|\mathcal{P}e_N + \mathcal{P}h_\varepsilon(U)e_N - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_N\| < \varepsilon$, namely

$$\|e_{-N} + \mathcal{P}h_\varepsilon(U)e_N - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_N\| < \varepsilon$$

But then we should also have

$$\|e_{-N} + \mathcal{P}h_\varepsilon(U)e_N - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_N\| \geq \|e_{-N} - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_N\| - \|\mathcal{P}h_\varepsilon(U)e_N\|$$

Hence

$$\varepsilon > \|e_{-N} + \mathcal{P}h_\varepsilon(U)e_N - \sum_{\alpha,\beta,h} c_{\alpha,\beta,h} S_\alpha S_\beta^* U^h e_N\| \geq 1 - \|\mathcal{P}h_\varepsilon(U)e_N\|$$

The conclusion is now got to if we can show that $\|\mathcal{P}h_\varepsilon(U)e_N\|$ is also as small as needed. But the norm $\|\mathcal{P}h_\varepsilon(U)e_N\|$ is easily computed in the Fourier transform of the canonical representation, where it takes the more workable form $(\int |h_\varepsilon(z^{-1})z^{-N}|^2 \mathrm{d}\mu(\mathrm{z}))^{\frac{1}{2}}$ and is accordingly smaller than $2\mu(\mathbb{T} \setminus C_\varepsilon)^{\frac{1}{2}} \leq 2\varepsilon^{\frac{1}{2}}$. The above inequality becomes absurd as soon as ε is taken small enough. □

As a straightforward consequence, we can immediately see that the class of the flip-flop in $\mathrm{Out}(\mathcal{Q}_2)$ do not coincide with any of the classes of the gauge automorphisms. In other terms, the automorphisms $\tilde{\lambda}_z \circ \tilde{\lambda}_f^{-1}$ are all outer, sending U in U^*. Furthermore, if we now denote by λ_{-1} the automorphism such that $\lambda_{-1}(S_2) := S_2$ and $\lambda_{-1}(U) := U^{-1} = U^*$, then the above result also applies to $\lambda_z := \lambda_{-1} \circ \tilde{\alpha}_z$, which are outer as well by the same reason. Also note that $\lambda_z(S_2) = zS_2$. More interestingly, if z is not 1, the corresponding λ_z yields a class in $\mathrm{Out}(\mathcal{Q}_2)$ other than the one of the flip-flop. To make sure this is true, we start by noting that λ_{-1} and $\tilde{\lambda}_f$ do not commute with each other. Even so, they do commute in $\mathrm{Out}(\mathcal{Q}_2)$, in that they even yield the same conjugacy class. Indeed, we have $\mathrm{ad}(U^*) \circ \lambda_{-1} = \tilde{\lambda}_f$, or equivalently $\tilde{\lambda}_f \circ \lambda_{-1} = \mathrm{Ad}(U^*)$, which in addition says that $\tilde{\lambda}_f \circ \lambda_{-1}$ has infinite order in $\mathrm{Aut}(\mathcal{Q}_2)$ while being the product of two automorphisms of order 2. From this our claim follows easily. For, if $\lambda_z \circ \tilde{\lambda}_f$ is an inner automorphism, the identity $\lambda_z \circ \tilde{\lambda}_f = \tilde{\lambda}_z \circ \lambda_{-1} \circ \tilde{\lambda}_f = \tilde{\lambda}_z \circ \mathrm{Ad}(U)$ implies at once that $\tilde{\lambda}_z$ is inner as well, which is possible for $z = 1$ only. However, the classes $[\lambda_z]$ and $[\tilde{\lambda}_f]$ do commute in $\mathrm{Out}(\mathcal{Q}_2)$, because $\lambda_z \circ \tilde{\lambda}_f \circ \lambda_z^{-1} \circ \tilde{\lambda}_f = \mathrm{Ad}(U^2)$. In order to prove that $\mathrm{Out}(\mathcal{Q}_2)$ is not abelian, we still need to sort out a new class of outer automorphisms. This will be done in the next sections.

3.5 Notable endomorphisms and automorphism classes

3.5.1 Endomorphisms and automorphisms α such that $\alpha(S_2) = S_2$

For any odd integer $2k + 1$, whether it be positive or negative, the pair (S_2, U^{2k+1}) still satisfies the two defining relations of $\mathcal{Q}_2$. This means that the map that takes S_2 to itself and U to U^{2k+1} extends to an endomorphism of $\mathcal{Q}_2$, which will be denoted by λ_{2k+1}. Trivially, this endomorphism extends the identity automorphism of $C^*(S_2)$. A slightly less obvious thing to note is that these endomorphisms cannot be obtained as extensions of endomorphisms of Ω_2. Indeed, $\lambda_{2k+1}(S_1) = \lambda_{2k+1}(US_2) = U^{2k+1}S_2$, and $U^{2k+1}S_2$ is not in Ω_2: if it were, we would also find that $S_1^*U^{2k+1}S_2 = S_2^*U^*UU^{2k}S_2 = S_2^*S_2U^k = U^k$ would be in Ω_2, which is not. In this way we get a class endomorphisms λ_{2k+1}, $k \in \mathbb{Z}$ with $\lambda_1 = \mathrm{id}$ and λ_{-1} being clearly an automorphism of order two. All these endomorphisms commute with one other and we have $\lambda_{2k+1} \circ \lambda_{2h+1} = \lambda_{(2k+1)(2h+1)}$ for any $k, h \in \mathbb{Z}$. Phrased differently, the set $\{\lambda_{2k+1} : k \in \mathbb{Z}\}$ is a semigroup of proper endomorphisms of $\mathcal{Q}_2$. One would like to know if the endomorphisms singled out above give the complete list of the endomorphisms of $\mathcal{Q}_2$ fixing S_2. In other words, the question is whether the set

$$\mathcal{U}_2 \doteq \{V \in U(\mathcal{Q}_2) \mid V^2S_2 = S_2V,\ S_2S_2^* + VS_2S_2^*V^* = 1\}$$

contains elements other than the U^{2k+1} with $k \in \mathbb{Z}$ above. As a matter of fact, answering this question in its full generality is not an easy task. An interesting if partial result does surface, though, as soon as we introduce an extra assumption. Going back to the endomorphisms λ_{2k+1}, we next show they are all proper apart from $\lambda_{\pm 1}$. The proof cannot be considered quite elementary, in that it uses the maximality of $C^*(U)$.

Proposition 3.5.1. *None of the endomorphisms λ_{2k+1} is surjective if $2k+1 \neq \pm 1$.*

Proof. Let $\mathfrak{A} \subset \mathcal{Q}_2$ the C^*-subalgebra of those $x \in \mathcal{Q}_2$ such that $\lambda_{2k+1}(x) \in C^*(U)$. We clearly have $C^*(U) \subset \mathfrak{A}$. Furthermore, as $\mathfrak{A}$ is an abelian subalgebra, we must also have $\mathfrak{A} = C^*(U)$. From this, it now follows that U is not in the range of λ_{2k+1}, for the restriction $\lambda_{2k+1} \restriction_{C^*(U)}$ is induced, at the spectrum level, by the map $\mathbb{T} \ni z \mapsto z^{2k+1} \in \mathbb{T}$. □

In addition, in the canonical representation the endomorphisms λ_{2k+1} cannot be implemented by any unitary $W \in \mathcal{B}(\ell_2(\mathbb{Z}))$. More precisely, we can state the following result.

Proposition 3.5.2. *Let $2k+1$ be an odd integer different from 1. Then there is no unitary V in $\mathcal{Q}_2$ such that $VS_2 = S_2V$ and $VUV^* = U^{2k+1}$.*

Proof. The proposition is easily proved by *reductio ad absurdum*. Let V be such a unitary as in the statement. From $VS_2e_0 = S_2Ve_0$ we deduce that Ve_0 is an eigenvector of S_2 with eigenvalue 1. Without loss of generality, we may assume that $Ve_0 = e_0$. Now,

$$Ve_h = VU^he_0 = U^{(2k+1)h}Ve_0 = U^{(2k+1)h}e_0 = e_{(2k+1)h},\ h = 0, 1, 2, \ldots$$

and, similarly,

$$Ve_{-h} = e_{-(2k+1)h},\ h = -1, -2, \ldots\ .$$

To conclude, it is now enough to observe that V is not surjective whenever $2k+1 \neq -1$, whereas the case of $2k+1 = -1$ leads to V being equal to $\mathcal{P}$, which does not belong to $\mathcal{Q}_2$. □

Rather than saying what the whole $\mathcal{U}_2$ is, we shall focus on its subset $\mathcal{U}_2 \cap C^*(U)$ instead, which is more easily dealt with. This task is accomplished by the next result.

Theorem 3.5.3. *The set $\mathcal{U}_2 \bigcap C^*(U)$ is exhausted by the odd powers of U, i.e. $\mathcal{U}_2 \bigcap C^*(U) = \{U^{2k+1} : k \in \mathbb{Z}\}$.*

Proof. Let $W \in \mathcal{U}_2 \bigcap C^*(U)$. Then there exists a function $f \in C(\mathbb{T}, \mathbb{T})$ such that $W = f(U)$. The condition $S_2W = W^2S_2$ can be rewritten as $S_2f(U) = f(U)^2S_2$. On the other hand, we also have $S_2f(U) = f(U^2)S_2$. Therefore $f(U^2)S_2 = f(U)^2S_2$, that is $f(U^2) = f(U)^2$. Accordingly, the function f must satisfy the functional equation $f(z^2) = f(z)^2$. Being continuous, our function f must be of the form $f(z) = z^n$ for some integer $n \in \mathbb{Z}$, see the Appendix. This means that $W = U^n$. If we also impose the condition on the ranges $S_2S_2^* + U^nS_2S_2^*U^{-n} = 1$, we finally find that n is forced to be an odd number, say $n = 2k+1$. □

The result obtained above can also be stated in terms of endomorphisms of $\mathcal{Q}_2$. With this in mind, we need to introduce a bit of notation. In particular, we denote by $\mathrm{End}_{C^*(S_2)}(\mathcal{Q}_2, C^*(U))$ the semigroup of those endomorphisms of $\mathcal{Q}_2$ that fix S_2 and leave $C^*(U)$ globally invariant.

Corollary 3.5.4. *The semigroup* $\mathrm{End}_{C^*(S_2)}(\mathcal{Q}_2, C^*(U))$ *identifies with* $\{\lambda_{2k+1} : k \in \mathbb{Z}\}$. *As a result, we also have*

$$\mathrm{Aut}_{C^*(S_2)}(\mathcal{Q}_2, C^*(U)) = \{id, \lambda_{-1}\} \cong \mathbb{Z}_2$$

The foregoing result might possibly be improved by dropping the hypothesis that our endomorphisms leave $C^*(U)$ globally invariant also. This is in fact a problem we are resolved to go back to elsewhere.

3.5.2 Automorphisms α such that $\alpha(U) = U$

In this section we study those endomorphisms and automorphisms $\Lambda \in \mathcal{Q}_2$ such that $\Lambda(U) = U$. This problem may be described in an equivalent way as determining the set

$$\mathcal{S}_2 := \{W \in Q_2 : W^*W = 1, WU = U^2W, WW^* + UWW^*U^* = 1\} .$$

We start with exhibiting some families of automorphisms. Given a function $f \in C(\mathbb{T}, \mathbb{T})$ we denote by β_f the automorphism of $\mathcal{Q}_2$ given by $\beta_f(U) = U$ and $\beta_f(S_2) = f(U)S_2$, which is well defined because the pair $(f(U)S_2, U)$ still satisfies the two defining relations of $\mathcal{Q}_2$. Note that $\beta_f \circ \beta_g = \beta_{f \cdot g}$ so that we obtain an abelian subgroup of $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$ and that a constant function $f(z) = e^{i\theta}$ gives back the gauge automorphism $\tilde{\alpha}_\theta$. Furthermore, we have the following result, which gives a sufficient condition on f for the corresponding β_f to be outer. As the condition is not at all restrictive, the correspondence $f \to \beta_f$, which is one to one, provides plenty of outer automorphisms.

Proposition 3.5.5. *If* $f \in C(\mathbb{T}, \mathbb{T})$ *is such that* $f(1) \neq 1$, *then* β_f *is an outer automorphism.*

Proof. If $V \in \mathcal{Q}_2$ is a unitary such that $\beta_f = \mathrm{ad}(V)$, then V commutes with U and therefore it is of the form $g(U)$ for some $g \in C(\mathbb{T}, \mathbb{T})$ by maximality of $C^*(U)$. The condition $\beta_f(S_2) = \mathrm{ad}(V)(S_2)$ yields the equation $f(U)S_2 = g(U)S_2g(U)^*$, that is $f(U)S_2 = g(U)g(U^2)^*S_2$. But then g and f satisfy the relation $f(z) = g(z)\overline{g(z^2)}$ for every $z \in \mathbb{T}$. In particular, the last equality says that $f(1) = g(1)\overline{g(1)} = 1$. □

However, the condition spotted above is not necessary. This will in turn result as a consequence of the following discussions. We will be first concerned with the problem as to whether an automorphism β_f may be equivalent in $\mathrm{Out}(\mathcal{Q}_2)$ to a gauge automorphism. If so, there exist $z_0 \in \mathbb{T}$ and $W \in \mathcal{U}(\mathcal{Q}_2)$ such that $WUW^* = U$ and $Wf(U)S_2W^* = z_0S_2$. As usual, the first relation says that $W = h(U)$ for some $h \in C(\mathbb{T}, \mathbb{T})$, which makes the second into $h(U)f(U)h(U^2)^* = z_0$, that is h satisfies the functional equation $h(z)f(z)\overline{h(z^2)} = z_0$. The latter says in particular that z_0 is just $f(1)$. We next show that there actually exist many continuous functions f such that there is no continuous h that satisfies

$$\overline{h(z)}h(z^2) = f(z)\overline{f(1)} := \Psi(z) . \tag{3.5.1}$$

Note that $\Psi(1) = 1$ and that there is no loss of generality if we assume $h(1) = 1$ as well. By evaluating (3.5.1) at $z = -1$ we find $h(-1) = 1$.

Remark 3.5.6. By density, the continuous solutions of the equation $\overline{h(z)}h(z^2) = \Psi(z)$ are completely determined by the values they take at the 2^n−th roots of unity. Furthermore, the value of such an h at a point z with $z^{2^n} = 1$ is simply given by the interesting formula $h(z) = \frac{1}{\prod_{k=0}^{n-1} \Psi(z^{2^k})}$. The latter is easily got to by induction starting from the relation $h(z^2) = h(z)\Psi(z)$.

Here is our result, which provides examples of β_f not equivalent with any of the gauge automorphisms. Let $f \in C(\mathbb{T}, \mathbb{T})$ be such that $f(e^{i\theta}) = 1$ for $0 \le \theta \le \pi$ and $f(e^{i\theta}) = -1$ for $\pi + \varepsilon \le \theta \le 2\pi - \varepsilon$ with $0 < \varepsilon \le \frac{\pi}{4}$, then we have the following.

Proposition 3.5.7. *If $f \in C(\mathbb{T}, \mathbb{T})$ is a function as above, then the associated β_f is not equivalent to gauge automorphisms.*

Proof. With the above notations, suppose that $h(z)$ is a solution of (3.5.1) such that $h(1) = h(-1) = 1$, which is not restrictive. Since $\Psi(i) = \Psi(e^{i\frac{\pi}{2}}) = 1$ and $h(i^2) = h(-1) = 1$, we immediately see that $h(i) = 1$. As $\Psi(z) = 1$ for $0 \le \theta \le \pi$, by using the functional equation we find that $h(e^{i\frac{\pi}{2^n}}) = 1$ for any $n \in \mathbb{N}$. Consider then $z = e^{5i\frac{\pi}{4}}$. We have $\Psi(e^{5i\frac{\pi}{4}}) = -1$ and $h(e^{2(5i\frac{\pi}{4})}) = h(i) = 1$, which in turn gives $h(e^{5i\frac{\pi}{4}}) = -1$. By induction we also see $h\big(e^{\frac{5\pi i}{2^{n+2}}}\big) = -1$. This proves that any solution h of the functional equation with f as in the statement cannot be continuous. □

We can now devote ourselves to answering the question as to whether $\mathrm{Out}(\mathcal{Q}_2)$ is abelian. It turns out that it is not. Our strategy is merely to show that automorphisms β_f corresponding to suitable functions f do not commute in $\mathrm{Out}(\mathcal{Q}_2)$ with the flip-flop. To begin with, if β_f does commute with $\tilde{\lambda}$ in $\mathrm{Out}(\mathcal{Q}_2)$, then there must exist a unitary $V \in \mathcal{Q}_2$ such that $\tilde{\lambda}_f \circ \beta_f \circ \tilde{\lambda}_f = \mathrm{ad}(V) \circ \beta_f$. Exactly as above, the unitary V is then a continuous function of U, say $V = h(U)$. In addition, we also have

$$f(U^*)S_2 = h(U)f(U)S_2h(U)^* = h(U)f(U)h(U^2)^*S_2$$

and so we find that f and h satisfy the equation $f(\bar{z}) = h(z)f(z)\overline{h(z^2)}$ for every $z \in \mathbb{T}$, which can finally be rewritten as $f(\bar{z})\overline{f(z)} = h(z)\overline{h(z^2)}$, to be understood as an equation satisfied by the unknown function h, with f being given instead. We next exhibit a wide range of continuous functions f for which the corresponding h does not exist. To state our result as clearly as possible, we fix some notation first. Let $f \in C(\mathbb{T}, \mathbb{T})$ be such that $f(e^{i\frac{9\pi}{8}}) = i$, and $f(z) = 1$ everywhere apart from a sufficiently small neighborhood of $z = e^{i\frac{9\pi}{8}}$.

Proposition 3.5.8. *If $f \in C(\mathbb{T}, \mathbb{T})$ is as above, then β_f does not commute with the flip-flop in* $\mathrm{Out}(\mathcal{Q}_2)$.

Proof. Repeat almost verbatim the same argument as in the foregoing proposition, now verifying that $h\big(e^{\frac{\pi i}{2^n}}\big) = 1$ first and then $h\big(e^{\frac{9\pi i}{2^{n+3}}}\big) = -1$. □

Notably, this also yields the announced result on $\mathrm{Out}(\mathcal{Q}_2)$.

Theorem 3.5.9. *The group* $\mathrm{Out}(\mathcal{Q}_2)$ *is not abelian.*

We end this section by proving that $\mathcal{S}_2$ is in fact exhausted by isometries of the form $f(U)S_2$, where f is a continuous function onto $\mathbb{T}$. This requires some preliminary work. First observe that given any $s \in \mathcal{S}_2$, a straightforward computation says that both s^*S_2 and s^*S_1 commute with U, but then by maximality of $C^*(U)$ we can rewrite them as $h(U)$ and $g(U)$ respectively, with h and g being continuous functions.

Lemma 3.5.10. *There exists a continuous function f such that $s = f(U)S_2$.*

Proof. We start with the equality $s^* = s^*(S_1S_1^* + S_2S_2^*) = (s^*S_1)S_1^* + (s^*S_2)S_2^*$, in which we substitute the above expressions. This leads to $s^* = g(U)S_1^* + h(U)S_2^*$, that is $s = S_1 g(U)^* + S_2 h(U)^* = US_2\overline{g(U)^*} + \overline{h(U^2)}^*S_2 = (Ug(U^2)^* + h(U^2)^*)S_2$. Therefore, our claim is true with $f(z) = z\overline{g(z^*)} + \overline{h(z^2)}$. □

Lemma 3.5.11. *With the notations set above, for every $z \in \mathbb{T}$ we have $|h(z)|^2 + |g(z)|^2 = 1$.*

Proof. It is enough to rewrite the equality $s^*s = 1$ in terms of h and g. □

Lemma 3.5.12. *With the notations set above, for every $z \in \mathbb{T}$ we have $zh(z)\overline{g(z)} + g(z)\overline{h(z)} = 0$.*

Proof. Once again it is enough to rewrite the equality $s^*Us = 0$, which is merely the orthogonality relation between s and Us, in terms of h and g. □

We are at last in a position to prove the main result on $\mathcal{S}_2$.

Theorem 3.5.13. *If $s \in \mathcal{S}_2$, then there exists a $f \in C(\mathbb{T}, \mathbb{T})$ such that $s = f(U)S_2$.*

Proof. At this stage, all we have to do is prove that $|f(z)|^2 = 1$. But

$$|f(z)|^2 = \left(\overline{g(z^2)}z + \overline{h(z^2)}(g(z^2)\overline{z} + h(z^2)\right) = 1 + z\overline{g(z^2)}h(z^2) + \bar{z}g(z^2)h(z^2) = 1$$

□

As an immediate consequence, we finally gain full information on $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$.

Theorem 3.5.14. *The equalities hold*

$$\mathrm{End}_{C^*(U)}(\mathcal{Q}_2) = \mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2) = \{\beta_f : f \in C(\mathbb{T}, \mathbb{T})\}$$

In particular, the semigroup $\mathrm{End}_{C^*(U)}(\mathcal{Q}_2)$ is actually a group isomorphic with $C(\mathbb{T}, \mathbb{T})$.

Remark 3.5.15. The bijective correspondence $f \leftrightarrow \beta_f$ is also a homeomorphism between $C(\mathbb{T}, \mathbb{T})$ equipped with the uniform convergence topology and $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$ endowed with the norm pointwise convergence.

We conclude this section proving that $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$ is in addition a maximal abelian subgroup of $\mathrm{Aut}(\mathcal{Q}_2)$.

Theorem 3.5.16. *The group* $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$ *is maximal abelian in* $\mathrm{Aut}(\mathcal{Q}_2)$.

Proof. We have to show that if $\alpha \in \mathrm{Aut}(\mathcal{Q}_2)$ commutes with any element of $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$ then α is itself an element of the latter group. Now, the equality $\alpha \circ \mathrm{ad}(U) = \mathrm{ad}(U) \circ \alpha$ gives $\mathrm{ad}(\alpha(U)) = \mathrm{ad}(U)$. Therefore, $\alpha(U) = zU$ for some $z \in \mathbb{T}$ by simplicity of $\mathcal{Q}_2$. The conclusion is then achieved if we show that actually $z = 1$. Exactly as above, we also have $\mathrm{ad}(\alpha(g(U))) = \mathrm{ad}(g(U))$ for any $g \in C(\mathbb{T}, \mathbb{T})$. Again, thanks to simplicity we see that $g(zU) = g(\alpha(U)) = \alpha(g(U)) = \lambda g(U)$ for some $\lambda \in \mathbb{T}$, possibly depending on g. In terms of functions we find the equality $g(zw) = \lambda_g g(w)$, which can hold true for any $g \in C(\mathbb{T}, \mathbb{T})$ only if $z = 1$. Indeed, when $z \neq 1$ is not a root of unity the characters w^n are the sole eigenfunctions of the unitary operator Φ_z acting on $L^2(\mathbb{T})$ as $(\Phi_z f)(w) := f(zw)$. Finally, the case of a z that is a root of unity is dealt with similarly. □

Remark 3.5.17. The findings above are worth comparing with a result obtained in [Cun80] that the group of automorphisms of $\mathcal{O}_2$ fixing the diagonal $\mathcal{D}_2$ is maximal abelian too.

The theorem also enables to thoroughly describe the automorphisms that send U to U^*, which have been shown to be automatically outer.

Theorem 3.5.18. *If α is an automorphism of $\mathcal{Q}_2$ such that $\alpha(U) = U^*$, then $\alpha(S_2) = f(U)S_2$ for a suitable $f \in C(\mathbb{T}, \mathbb{T})$.*

Proof. Just apply the former result to $\tilde{\lambda}_f \circ \alpha$. □

Finally, the automorphisms β_f can also be characterized in terms of the Cuntz-Takesaki generalized correspondence we discussed at the end of Section 4.1. Indeed, they turn out to be precisely those $\Lambda \in \mathrm{End}(\mathcal{Q}_2)$ for which the corresponding W equals 1. For $W = 1$ we find in fact the equality $S_2 u U u^* = U^2 S_2 = S_2 U$, whence $uUu^* = U$. Therefore by maximality there exists a function $f \in C(\mathbb{T}, \mathbb{T})$ such that $u = f(U)$, that is $\Lambda = \beta_f$.

3.5.3 Automorphisms α such that $\alpha(U) = zU$

The following discussion addresses the problem of studying those automorphisms Λ of $\mathcal{Q}_2$ such that $\Lambda(U) = zU$, with $z \in \mathbb{T}$. We start tackling the problem by defining two operators acting on $\ell_2(\mathbb{Z})$. The first is the isometry S_z', which is given by $S_z' e_k \doteq z^k e_{2k}$. The second is the unitary U_z, which is given by $U_z e_k \doteq z^k e_k$. The following commutation relations are both easily verified:

- $U_z U = zUU_z$
- $U_z S_2 = S_z' U_z$

The first relation can also be rewritten as $\mathrm{ad}(U_z)(U) = zU$. We caution the reader that at this level $\mathrm{ad}(U_z)$ makes sense as an automorphism of $B(\ell_2(\mathbb{Z}))$ only, because we do not know yet whether U_z sits in $\mathcal{Q}_2$. If it does, the first relation says, inter alia, that $\mathcal{Q}_2$

also contains a copy of the noncommutative torus $\mathcal{A}_z$ in a rather explicit way, which is worth mentioning. In order to decide what values of z do give a unitary U_z belonging to $\mathcal{Q}_2$, the first thing to note is that if U_z is in $\mathcal{Q}_2$, then it must be in the diagonal subalgebra $\mathcal{D}_2$, as shown in the following lemma.

Lemma 3.5.19. *If U_z is in $\mathcal{Q}_2$, then $U_z \in \mathcal{D}_2$.*

Proof. A straightforward application of the equality $\mathcal{D}_2' \cap \mathcal{Q}_2 = \mathcal{D}_2$, as U_z is in $\mathcal{D}_2' = \ell_\infty(\mathbb{Z})$. □

The second thing to note is that the unitary representation $\mathbb{T} \ni z \mapsto U_z \in \mathcal{U}(B(\ell_2(\mathbb{Z})))$ is only strongly continuous. This implies that not every U_z is an element of $\mathcal{Q}_2$. For the representation $z \mapsto U_z$ is only strongly continuous, which means the set $\{U_z\}_{z\in\mathbb{T}}$ is not separable with respect to the norm topology, whereas $\mathcal{Q}_2$ obviously is.

Proposition 3.5.20. *If $z \in \mathbb{T}$ satisfies $z^{2^n} = 1$ for some natural number n, then U_z is in $\mathcal{D}_2$.*

Proof. Obviously only primitive roots have to be dealt with. But for such roots, say $z = e^{i2\pi/2^k}$, the unitary U_z may in fact be identified to the sum $\sum_{j=0}^{2^{k-1}} z^j P_j$, where the projection P_j belongs to $\mathcal{D}_2$, being more explicitly given by $P_{i_1 i_2 \ldots i_k}$, where the multi-index $(i_1, i_2, \ldots, i_k) \in \{1,2\}^k$ is the j-th with respect to the lexicographic order in which $2 < 1$ and the multi-index itself is read from right to left. □

The automorphisms obtained above are of course of finite order. More precisely, the order of $\mathrm{ad}(U_z)$ is just the same as the order of the corresponding z. In other words, what we know is that the automorphism of $C^*(U)$ induced by the rotation on $\mathbb{T}$ by a 2^n-th root of unity extends to an inner automorphism of $\mathcal{Q}_2$, whose order is still finite being just 2^n. Due to the lack of norm continuity of the representation $z \to U_z$, though, the case of a general z is out of the reach of the foregoing proposition and must needs be treated separately with different techniques. For this we need a preliminary lemma whose content should be well known. Nevertheless, we do include a proof for the sake of completeness.

Lemma 3.5.21. *Any projection $P \in \mathcal{D}_2$ is in the linear algebraic span of $\{S_\alpha S_\alpha^*\}_{\alpha\in W}$.*

Proof. It is convenient to realize $\mathcal{D}_2$ as the concrete C^*-algebra $C(K)$, with the spectrum K being given by the Tychonoff product $\{1,2\}^{\mathbb{N}}$. If we do so, the projections of $\mathcal{D}_2$ are immediately seen to identify with the characteristic functions of the clopens of K, and these are clearly the cylinder sets in the product space. The conclusion now follows noting that for any multi-index $\alpha \in W_2$ the characteristic function of a cylinder $C_\alpha = \{x \in K : x(k) = \alpha_k \text{ for any } k = 0, 1, \ldots l(\alpha)\}$ corresponds indeed to $S_\alpha S_\alpha^*$. □

The lemma above allows us to prove that no root of unity z may yield a U_z that belongs to $\mathcal{Q}_2$ apart from the dyadic ones.

Proposition 3.5.22. *Suppose that $z \in \mathbb{T}$ is a root of unity such that $z^{2^k} \neq 1$ for all natural numbers k. Then U_z is not in $\mathcal{D}_2$.*

Proof. Suppose that z is an nth root of unity. In particular, the operator U_z has finite order, and so $U_z = \sum_{k=1}^n d_k P_k$ for some $P_k \in \mathcal{D}_2$, $d_k \in \mathbb{T}$ thanks to the continuous functional calculus. Furthermore, by Lemma 3.5.21 we have that each projection P_k is the linear combination of some $S_\alpha S_\alpha^*$. By using the equality $S_1S_1^* + S_2S_2^* = 1$, we see that there is no loss of generality if we also suppose that

$$U_z = \sum_{|\alpha|=h} c_\alpha S_\alpha S_\alpha^* \quad c_\alpha \in \mathbb{T} .$$

On the one hand we know that $U_z e_0 = e_0$, but on the other $U_z e_0 = c_{(2,\dots,2)} e_0$. This means that $c_{(2,\dots,2)} = 1$. Since $e_{2^h l}$ is in the range of the projection $S_2^h(S_2^h)^*$, we must also have the equality $U_z e_{2^h l} = c_{(2,\dots,2)} e_{2^h l} = e_{2^h l}$. But by its very definition $U_z e_{2^h l}$ is $z^{2^h l} e_{2^h l}$, therefore $z^{2^h l} = 1$ for every $l \in \mathbb{Z}$. Taking $l = 1$, we finally see that $z^{2^h} = 1$. □

We are finally left with the task of showing that any U_z coming from a z that is not a root of unity cannot sit in the diagonal subalgebra $\mathcal{D}_2$, which is proved in the next proposition.

Proposition 3.5.23. *The unitary U_z is not in $\mathcal{D}_2$ if $z \in \mathbb{T}$ is not a root of unity.*

Proof. We shall argue by contradiction. If U_z did belong to $\mathcal{D}_2$, then there would exist a finite subset $F \subset W_2$ of multi-indeces such that

$$\|U_z - \sum_{\alpha \in F} c_\alpha S_\alpha S_\alpha^*\| < \varepsilon$$

with $\sum_{\alpha\in F} c_\alpha S_\alpha S_\alpha^*$ being unitary as well. In particular, we would have $\|z^k e_k - \sum_{\alpha\in F} c_\alpha S_\alpha S_\alpha^* e_k\| < \varepsilon$ for every integer k. As usual, no loss of generality occurs if we also assume that the α's are all of the same length, say h. We have that

$$\|e_0 - \sum_{\alpha\in F} c_\alpha S_\alpha S_\alpha^* e_0\| = |1 - c_{2,\dots,2}| < \varepsilon ,$$

which in turn implies

$$\|z^{2^h l} e_{2^h l} - \sum_{\alpha\in F} c_\alpha S_\alpha S_\alpha^* e_{2^h l}\| = |z^{2^h l} - c_{2,\dots,2}| < \varepsilon ,$$

Now, if z is not a root of unity, then z^{2^h} is not a root of unity and thus $\{(z^{2^h})^l\}_{l\in\mathbb{Z}}$ is dense in $\mathbb{T}$, and thus an absurd is finally arrived at. □

To sum up, we have proved the following result.

Theorem 3.5.24. *Let $z \in \mathbb{T}$. Then $U_z \in \mathcal{D}_2$ if and only if z is a root of unity of order a power of 2.*

For those $z \in \mathbb{T}$ such that U_z does lie in $\mathcal{Q}_2$ we can say a bit more.

Proposition 3.5.25. *Let $z \in \mathbb{T}$ such that $U_z \in \mathcal{Q}_2$ and $\alpha \in Aut(\mathcal{Q}_2)$ such that $\alpha(U) = zU$. Then there exists a $f \in C(\mathbb{T}, \mathbb{T})$ such that $\alpha(S_2) = f(zU)S'_z$.*

Proof. By its very definition $\mathrm{ad}(U_{z^{-1}}) \circ \alpha(U) = U$. Therefore, we must have $\mathrm{ad}(U_{z^{-1}}) \circ \alpha = \beta_f$ for some $f \in C(\mathbb{T}, \mathbb{T})$. But then $f(U)S_2 = \beta_f(S_2) = U_{z^{-1}}\alpha(S_2)U_z$, i.e. $\alpha(S_2) = U_z f(U)S_2 U_{z^{-1}} = f(zU)U_z S_2 U_{z^{-1}} = f(zU)S'_z$. □

Remark 3.5.26. We have already seen that if $U_z \in \mathcal{Q}_2$ then $S'_z \in \mathcal{Q}_2$. The converse, too, is true. In fact, one can easily observe that $U_z = S_2^* S'_z$ hence the claim follows. In particular, whenever U_z is not in $\mathcal{Q}_2$, the corresponding $\mathrm{ad}(U_z)$ understood as an automorphism of the whole $B(\ell_2(\mathbb{Z}))$ does not even leave $\mathcal{Q}_2$ globally invariant.

3.6 The functional equation $f(z^2) = f(z)^2$ on the torus

This is devoted to solving the functional equation $f(z^2) = f(z)^2$, of which we made an intensive use in the previous sections.

Proposition 3.6.1. *Let f be a continuous function from $\mathbb{T}$ to $\mathbb{T}$ such that $f(z^2) = f(z)^2$ for every $z \in \mathbb{T}$. There there exists a unique $n \in \mathbb{Z}$ such that $f(z) = z^n$.*

Proof. By compactness of $\mathbb{T}$ along with continuity of f, the winding number of f is a well-defined integer $n \in \mathbb{Z}$, for the details see e.g. Arveson's book [Arv06, Chapter 4, pp 114-115]. As a consequence, our function f must come from a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(e^{i\theta}) = e^{ig(\theta)}$ with $g(0) = 0$ and $g(2\pi) = 2n\pi$. Moroever, such a g can be assumed to be 2π periodic (up to multiples of 2π) and continuous in the open interval $(0, 2\pi)$. The functional equation can then be rewritten in terms of g as $g(2\theta) = 2g(\theta)$ (mod 2π). The proof is complete if we show that $g(\theta) = n\theta$ for every $\theta \in (0, 2\pi)$. By continuity, it is enough to prove this only on a dense subset of the interval $(0, 2\pi)$. To this aim, note that we already know $g\left(\frac{\pi}{2^k}\right) = n\frac{\pi}{2^k}$ for every $k \in \mathbb{N}$ by virtue of the equation itself. Now let ε be a real number with $0 \leq \varepsilon < \pi$. Then we have the chain of equalities $g(2\varepsilon) = g(2(\pi + \varepsilon)) = 2g(\pi + \varepsilon) + 2k_\varepsilon\pi$, where the first equality is due to 2π-periodicity and the second to the functional equation satisfied by g. By continuity, we can also say that k_ε does not depend on ε. If we then compute the equalities at $\varepsilon = 0$, we can easily see what k is. Indeed, we find $0 = 2n\pi + 2k\pi$, that is $k = -n$. Therefore our relation take the form $g(2\varepsilon) = 2g(\pi + \varepsilon) - 2n\pi$. If we now choose $\varepsilon = \frac{\pi}{2^k}$, we see that $g\left(\pi + \frac{\pi}{2^k}\right) = n\left(\pi + \frac{\pi}{2^k}\right)$, as wished. It is now clear how to go on inductively in order to show that $g(\theta) = n\theta$ for those θ corresponding to the midpoints of the intervals obtained by halving $[\pi, 2\pi]$ repeatedly, that is the dyadic rationals up to dividing by π. □

The above proposition can actually be regarded as a one-variable description of the characters of the one-dimensional torus. It is worth pointing out, though, that it no longer holds true as soon as $\mathbb{T}$ is replaced by the additive group $\mathbb{R}$. In other words, there do exist continuous functions $f : \mathbb{R} \to \mathbb{T}$ such that $f(2x) = f(x)^2$ other than

$f_t(x) := e^{itx}$, which are obtained by exponentiating non-linear continuous functions $g : \mathbb{R} \to \mathbb{R}$ such that $g(2x) = 2g(x)$ for every $x \in \mathbb{R}$. However, any such g cannot be everywhere differentiable with continuous derivative at 0.

Chapter 4

The HOMFLYPT polynomial and the oriented Thompson group $\vec{F}$

Recently Jones [Jon14] defined two ways of associating links to elements of the Thompson group F, roughly one designed for real values the variable A in the Kauffman bracket, one for A being a root of unity. We call the first one the *real procedure*, whereas the second will be called the *complex procedure*. These procedures yield un-oriented links, however the links associated to the elements of the oriented Thompson group $\vec{F} \leq F$ (also introduced by Jones) have a natural orientation. In [AC16b], by using elementary techiniques the author and Roberto Conti produced some unitary representations of $\vec{F}$ associated with the Homfly polynomial and related to the real construction. In [ACJ16] the author, Roberto Conti, and Vaughan F. R. Jones, produced unitary representations of $\vec{F}$ associated with the Homflypt polynomial and the complex procedure. In the latter paper more powerful techniques developed by Jones were used, see [Jon16]. We mention that this results can be naturally generalized, and the case of the Homflypt polynomial was just a particular application of these techniques. The objective of this chapter is to give a self-contained treatment of the results contained in [AC16b].

4.1 Preliminaries: some definitions and examples

This section is devoted to introducing the definition of the oriented Thompson group $\vec{F}$ and recalling the Jones' *real procedure* for producing oriented links out of it.

In the first place (see [CFP96]) we recall some definitions of the Thompson group F. The Thompson group F can be defined by the following finite presentation

$$\langle x_0, x_1 \mid x_2x_1 = x_1x_3, x_3x_1 = x_1x_4 \rangle,$$

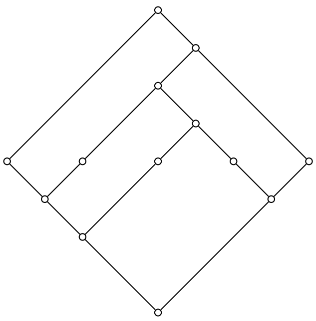
where $x_n := x_0^{1-n}x_1x_0^{n-1}$ for $n \geq 2$. In an alternative picture, F can be seen as a particular subgroup of the group of homeomorphisms of the interval $[0,1]$. Indeed, it is generated by the following homeomorphisms (see [CFP96] for further details)

$$x_0 = \begin{cases} 2t & 0 \le t \le \frac{1}{4} \\ t + \frac{1}{4} & \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{t}{2} + \frac{1}{2} & \frac{1}{2} \le t \le 1 \end{cases} \qquad x_1 = \begin{cases} t & 0 \le t \le \frac{1}{2} \\ 2t - \frac{1}{2} & \frac{1}{2} \le t \le \frac{5}{8} \\ t + \frac{1}{8} & \frac{5}{8} \le t \le \frac{3}{4} \\ \frac{t}{2} + \frac{1}{2} & \frac{3}{4} \le t \le 1 \end{cases}$$

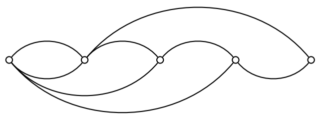
An equivalent description is the following, [Bel07]. One can define standard dyadic intervals, namely those whose endpoints are $\frac{k}{2^n}$ and $\frac{k+1}{2^n}$ for $n, k \in \mathbb{N}$. Any finite partition of the interval $[0,1]$ made with standard dyadic intervals is called a dyadic subdivision. Given two dyadic subdivision $\mathcal{A}$ and $\mathcal{B}$ with the same cardinality, it is possible to define a homeomorphism $f_{\mathcal{A},\mathcal{B}} : [0,1] \to [0,1]$ which maps linearly each interval of $\mathcal{A}$ onto the corresponding interval of $\mathcal{B}$. The maps $f_{\mathcal{A},\mathcal{B}}$ form the group F. This characterization of the Thompson group has the following graphical description. Set $\mathcal{T} := \bigcup_n \mathcal{T}_n$ the set of rooted planar binary trees. With $T \in \mathcal{T}$, we denote by $\partial T = \{f_1, \ldots, f_n\}$ the set of leaves of T. Of course, $\mathcal{T}_n := \{T \in \mathbb{T} \mid |\partial T| = n\}$. Denote by $\mathcal{T}_\partial^2 := \mathcal{T} \times_\partial \mathcal{T}$ the set of matched pairs of trees (T_+, T_-), i.e. such that $|\partial T_+| = |\partial T_-|$. We also say that any such pair (T_+, T_-) is bifurcating. To any leaf of a binary tree it is associated a standard dyadic interval (see [Bel07, p. 13]), thus a pair of trees can be used to determine an element of F. Therefore, there is a map $\mathcal{T}_\partial^2 \to F$, $(T_+, T_-) \mapsto g(T_+, T_-)$. Indeed, it is surjective but not injective. A cheap way to see this is to realize that any such pair with $T_+ = T_-$ gives rise to the identity element of F. Moreover, it holds $g(T_-, T_+) = g(T_+, T_-)^{-1}$ and if $g(T_+, T_-) = g(T'_+, T'_-)$ it is possible to connect the two pairs by a sequence of addition/deletion of opposite carets.

Jones introduced a procedure that associates alternating links (up to linked unknots) to elements of R. Thompson group F, [Jon14, p. 18-19]. We recall the procedure with an example.

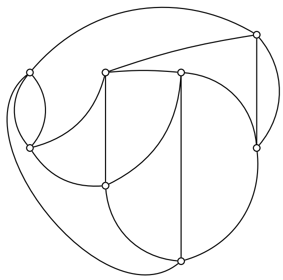
Example 4.1.1. Consider an element of the Thompson group associated with a pair of binary trees T_+ and T_- with n-leaves, for example

$\frac{T_+}{T_-} =$ 

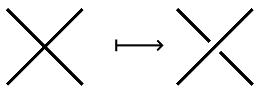
Suppose that the trees $T_\pm$ are on the plane $\mathbb{R}^2$, with leaves on the points $(1/2, 0)$, $(3/2, 0)$, ..., $((2n-1)/2, 0)$, T_+ being in the upper-half plane and T_- in the lower-half plane. The vertices of the graph $\Gamma(T_+, T_-)$ are $(0,0)$, $(1,0)$, ..., $(n-1, 0)$. The edges correspond to certain edges of the trees, namely those sloping up from left to right (here called WN edges) for the upper tree and those sloping down from left to right (called WS edges) for the bottom tree. Starting from the right-most vertex (the one with coordinates $(n-1,0)$), we draw a curve passing once transversally through the edges of the trees we mentioned. This is how the Γ-graph looks like in our example

$\Gamma(T_+, T_-) =$ 

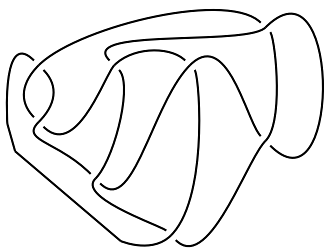
The next step is to draw the medial graph of $\Gamma(T_+, T_-)$.

$\Phi(\Gamma(T_+, T_-)) =$ 

The last and final step is to substitute any 4-valent vertex according to following rule



This is the associated alternating link $L(T_+, T_-)$



Let G be a graph. For any $k \in \mathbb{N}$, a proper vertex coloring is a map $\phi : V(G) := \{$ vertices of $G\} \to \{1, \ldots, k\}$ such that $\phi(v_1) \neq \phi(v_2)$ whenever the vertices v_1 and v_2 are connected. The Chromatic polynomial $\mathrm{Chr}_G(x)$ is a polynomial such that the evaluation at $k \in \mathbb{N}$ gives the number of its proper vertex colorings with k colours. In [Jon14, Section 5, p. 30], Jones associates to any element defined the oriented Thompson group as

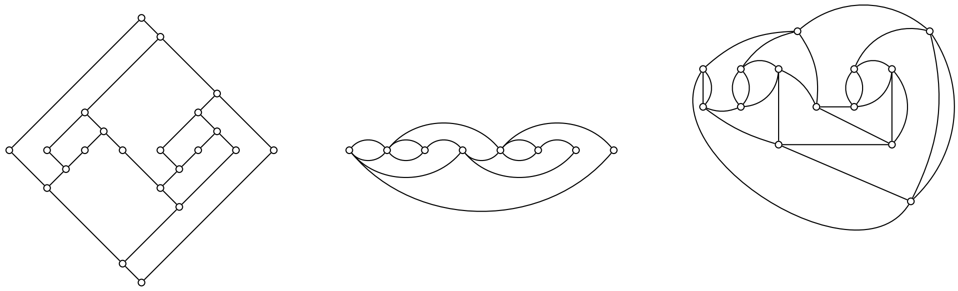
$$\vec{F} := \{g \in F \mid \mathrm{Chr}_{\Gamma(g)}(2) = 2\} .$$

In other words this is the subgroup of F made of elements whose Γ-graph is bipartite. Golan and Sapir proved that this group is finitely generated exhibiting some generators

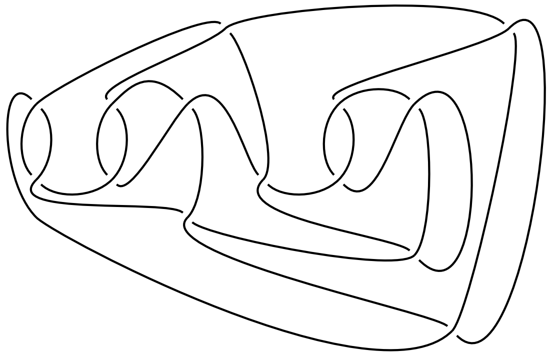
([GS15], Lemma 4.6), namely x_0x_1, x_1x_2 and x_2x_3. Moreover, using also the fact that these elements satisfy the defining relations of the generators of F_3 they actually showed that indeed $\vec{F}$ is isomorphic to the Thompson group F_3 [GS15, Lemma 4.7].

Using $\Gamma(g)$, Jones ([Jon14]) discovered a way to associate an oriented link to any element of the Jones-Thompson group. We briefly outline this procedure in the following example.

Example 4.1.2. Being $\vec{F}$ a subgroup of F, any element of $\vec{F}$ can be represented by a pair of trees. The first part of the procedure is the same as the one described earlier for any element of F. So we repeat the steps withouth further explanations. Consider the following element of $\vec{F}$, the associated Γ-graph and the corresponding medial graph

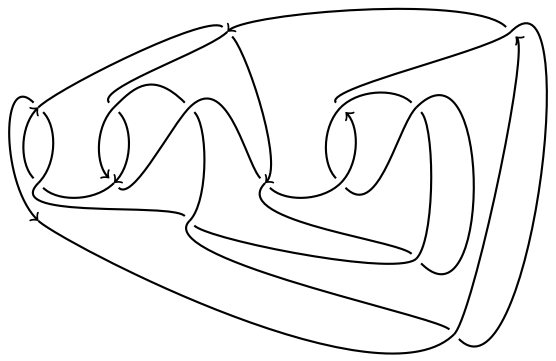


Indeed, this is an element of $\vec{F}$ as it can be easily seen that the associated Γ-graph is bipartite. This is the corresponding un-oriented link diagram



Denote by $\{+,-\}$ the two colours. The graph $\Gamma(T_+,T_-)$ has only two colourings: one in which the first vertex associated to the colour $+$ and one in which it is associated to the colour $-$. Consider the first couloring. The regions of the diagram of the un-oriented link $L(T_+,T_-)$ can be two coloured in black and white (we choose as a convention to colour the outer region in white). Each vertex of the Γ graphs sits precisely in one of the n black regions and its sign determine an orientation of the boundary ($+$ =counterclockwise,

$-$ =clockwise). The procedure yields the following link



4.2 Main result: the Homflypt polonomial as a positive type function

We recall that the HOMFLY polynomial $P_{\vec{L}}(\alpha, z)$ satisfies the following properties

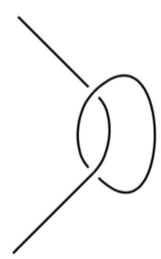
$$\begin{cases} \alpha P_{\nwarrow\!\!\!\!\nearrow}(\alpha, z) - \alpha^{-1} P_{\nearrow\!\!\!\!\nwarrow}(\alpha, z) - z P_{\uparrow\uparrow}(\alpha, z) = 0 \\ P_{\vec{L}_1 \cup \vec{L}_2}(\alpha, z) = \frac{\alpha - \alpha^{-1}}{z} P_{\vec{L}_1}(\alpha, z) P_{\vec{L}_2}(\alpha, z) \\ P_O = 1 \end{cases}$$

(actually it is defined by the first and the third equations). Our aim is to determine for which specializations of the involved variables this construction yields a function of positive type on $\vec{F}$ out of the HOMFLY polynomial.

We begin our analysis with some preliminary lemmas. We keep the same notation and terminology adopted in [Jon14] (cf. [AC15, AC16]). The first result is a slight variation of an argument already discussed in [Jon14] and we omit the easy proof.

Lemma 4.2.1. *(cf.* [Jon14, p.19] [AC15, Prop. 5]*) Let (T_+, T_-) be a pair of bifurcating trees with the same number of leaves. Consider another pair of (T'_+, T'_-) of such trees obtained from (T_+, T_-) by adding a pair of opposing carets. Then, the link $L(T'_+, T'_-)$ is obtained from $L(T_+, T_-)$ by the addition of a linked unknot.*

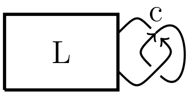
The following picture provides a self-explanatory diagrammatic illustration of the concept of linked unknot.

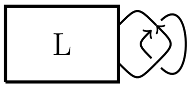


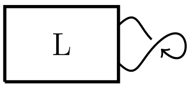
Lemma 4.2.2. *Let (T_+, T_-) be a pair of bifurcating trees with the same number of leaves such that $g(T_+, T_-) \in \vec{F}$. Consider another pair (T_+', T_-') of trees obtained by adding a pair of opposing carets. Then,*

$$P_{\vec{L}(T_+', T_-')}(\alpha, z) = \alpha^{-1}\left[\frac{1-\alpha^{-2}}{z} + z\right] P_{\vec{L}(T_+, T_-)}(\alpha, z) \ .$$

Proof. With the above notation consider the links $\vec{L} = \vec{L}(T_+, T_-)$, $\vec{L}_1 = \vec{L}(T_1', T_2')$, and $\vec{L}_2$, $\vec{L}_3$ obtained by applying the skein relation at the crossing c

$\vec{L}_1 =$ 

$\vec{L}_2 =$ 

$\vec{L}_3 =$ 

Using the skein relation we obtain

$$\begin{aligned} 0 &= \alpha P_{\vec{L}_1}(\alpha, z) - \alpha^{-1} P_{\vec{L}_2}(\alpha, z) - z P_{\vec{L}_3}(\alpha, z) \\ &= \alpha P_{\vec{L}_1}(\alpha, z) - \frac{\alpha - \alpha^{-1}}{z}\alpha^{-1} P_{\vec{L}}(\alpha, z) - z P_{\vec{L}}(\alpha, z) \end{aligned}$$

and thus we get the thesis. □

We define the modified HOMFLY function as

$$P_g(\alpha, z) := \left\{\alpha^{-1}\left[\frac{1-\alpha^{-2}}{z} + z\right]\right\}^{-n+1} P_{\vec{L}(T_+, T_-)}(\alpha, z) \ , \qquad g \in \vec{F}$$

where $g = g(T_+, T_-)$ for a pair of bifurcating trees with n leaves.

Next we need to state few more preliminary lemmas.

Lemma 4.2.3. *Let $g = g(T_+, T_-) \in \vec{F}$ be the element determined by a pair of trees (T_+, T_-) with n leaves. Then the writhe number of the link $\vec{L}(T_+, T_-)$ associated to g is given by* $\mathrm{wr}(\vec{L}(T_+, T_-)) = 2(n-1)$.

Proof. In the first place we recall that for each element in $\vec{F}$ the associated Γ-graph admits a 2-colouring (the colours will be denoted by $+$ and $-$). Moreover, each crossing in $L(T_+, T_-)$ corresponds to an edge of $\Gamma(T_+, T_-)$ and the orientation of the crossing depends on the colours of the two vertices connected by this edge. Now, consider an edge with left vertex coloured with $+$ and right vertex with $-$, then the corresponding crossing in the Γ-graph is $\nwarrow\!\!\!\!\!\!\nearrow$. On the other hand, if the left vertex has colour $-$ and the right vertex has colour $+$, then the associated crossing is again of the same type $\nwarrow\!\!\!\!\!\!\nearrow$. Finally observe that the Γ-graph has precisely $2(n-1)$ edges, which is also the number of crossings of the associated link diagram. □

Let $g(T_+, T_-)$ be an element of $\vec{F}$, where T_+, T_- are trees with n leaves. As already mentioned, the graph $\Gamma(T_+, T_-)$ admits only two 2-colourings. Consider the colouring for which the first vertex on the left has colour $+$. We denote by n_+ (resp. n_-) the number of vertices of $\Gamma(T_+, T_-)$ labeled with colour $+$ (resp. $-$) and by $\mathrm{rot}(\vec{L})$ the rotation number of the link $\vec{L}$ (cf. [Jon89, Def. 1.10]). We recall that the rotation number of a link $\vec{L}$ can be computed in the following way. Given a diagram of $\vec{L}$ resolve the crossing $\nearrow\!\!\!\!\nwarrow$ and $\nwarrow\!\!\!\!\nearrow$ as $\upuparrows$. Let m be the number of clockwise oriented loops, n be the number of counter-clockwise oriented loops, then $\mathrm{rot}(\vec{L}) = n - m$.

Lemma 4.2.4. *Let $g(T_+, T_-) \in \vec{F}$. We have that* $\mathrm{rot}(\vec{L}(T_+, T_-)) = n_+ - n_-$.

Proof. Since the rotation number may computed resolving crossing $\nearrow\!\!\!\!\nwarrow$ as $\upuparrows$, the number rot is the difference between the number of regions in a shaded diagram whose boundary is counter-clockwise oriented and the number of regions whose boundary is clockwise oriented. The first (resp. second) number coincides with n_+ (resp. n_-). $\square$

Using Lemma 4.2.1 we may also (well-)define the function $c(g) := c(L(T_+, T_-)) - n$ where $g = g(T_+, T_-) \in \vec{F}$ and (T_+, T_-) is a pair of trees with n leaves.

Lemma 4.2.5. *For any $g = g(T_+, T_-) \in \vec{F}$ for a pair of trees (T_+, T_-) with n leaves, we have that $c(g) \in -2\mathbb{N}$.*

Proof. With the above notations denote by $\vec{L}$ the link $\vec{L}(T_+, T_-)$. Let $b(\vec{L})$ be the number of black regions in the shaded diagram of $\vec{L}$. Since we are only interested in the number of the components of the link, after changing some crossings whenever necessary we get a new link $\vec{L}'$ which is trivial. We observe that $b(\vec{L}) = b(\vec{L}')$. We have that

$$\begin{aligned}(-1)^{c(g)} &= (-1)^{c(\vec{L})-n} = (-1)^{c(\vec{L})}(-1)^n = \\ &= (-1)^{c(\vec{L}')}(-1)^n = (-1)^{\mathrm{rot}(\vec{L}')}(-1)^n = \\ &= (-1)^{b(\vec{L})}(-1)^n = (-1)^n(-1)^n = 1\end{aligned}$$

where we used the fact that the rotation number may be computed resolving each crossing $\nearrow\!\!\!\!\nwarrow$ and $\nwarrow\!\!\!\!\nearrow$ as $\upuparrows$, and that the number of black regions in the shaded diagram is equal to the number of the vertices in the face graph. $\square$

Now we recall the formula for the partition function representation of the HOMFLY polynomial (introduced by Jones in [Jon89], also described in [Jae93]). Let $\vec{L}$ be an oriented link diagram. For $k \in \mathbb{N}$, the elements in $\Theta_k := \{1, \ldots, k\}$ are called colours. The functions $\tau : E(\vec{L}) \to \Theta_k$ are called states. Consider a 4-valent vertex with colours i and j as inputs, h and l as outputs. We define the following weights $w_\pm : \Theta_k \times \Theta_k \times$

$\Theta_k \times \Theta_k \to \mathbb{C}$ for positive and negative crossings, respectively, as

$$w_+(i,j;h,l) := \begin{cases} q-q^{-1} & \text{if } i<j,\ i=h,\ j=l \\ 1 & \text{if } i=l,\ j=h,\ i\neq j \\ q & \text{if } i=j=h=l \\ 0 & \text{otherwise} \end{cases}$$

$$w_-(i,j;h,l) := \begin{cases} q^{-1}-q & \text{if } i>j,\ i=h,\ j=l \\ 1 & \text{if } i=l,\ j=h,\ i\neq j \\ q^{-1} & \text{if } i=j=h=l \\ 0 & \text{otherwise} \end{cases}$$

Consider the partition function given by

$$Z_{\vec{L}}(q,k) := \frac{q-q^{-1}}{q^k-q^{-k}} q^{-k\mathrm{wr}(\vec{L})} q^{-(k+1)\mathrm{rot}(\vec{L})} \sum_{\tau} \prod_{x\in V(\vec{L})} w(\cdot) q^{2s(\vec{L},\tau)} ,$$

where the sum runs over the state functions, $w(\cdot)$ denotes the appropriate weight function, namely $w_\pm$ if the vertex is a positive/negative crossing, $\mathrm{rot}(\vec{L})$ is the rotation number of $\vec{L}$ and $s(\vec{L},\tau)$ is defined as follows: given state τ and $i \in \{1,\dots,k\}$, denote by $\vec{L}_i$ the link obtained as $\tau^{-1}(i)$, then $s(\vec{L},\tau) = \sum_{i=1}^{q} i \ \mathrm{rot}(\vec{L}_i)$ (if $\tau^{-1}(i) = \emptyset$, set $\mathrm{rot}(\vec{L}_i) = 0$). It can be proved that the above partition function satisfies the skein relation

$$q^k Z_{\nearrow\!\!\!\!\nwarrow}(q,k) - q^{-k} Z_{\nwarrow\!\!\!\!\nearrow}(q,k) = (q-q^{-1}) Z_{\uparrow\uparrow}(q,k)$$

and thus coincides with the Homfly polynomial $P_{\vec{L}}(q^k, q-q^{-1})$.

As observed in the proof of Lemma 4.2.3, the links considered here will only have crossings of type $\nearrow\!\!\!\!\nwarrow$. Accordingly we will only use the weight function $w_+(\cdot)$. Also notice that the weight w_+ is invariant under a switch of the upper and lower indices.

Using Jones's partition function model for the HOMFLY polynomial, we are now ready to state our main result.

Theorem 4.2.6. *For $q \in \mathbb{R} \setminus \{\pm 1, 0\}$ and k a positive integer, the evaluation of the modified HOMFLY function $g \mapsto P_g(q^k, q-q^{-1})$ is of positive type on $\vec{F}$.*

Proof. Without loss of generality we can suppose that the $g_i = g(T_+^i, T_-)$ where $\{T_+^i\}_{i=1}^r$ and T_- are trees with n leaves such that $g_i g_j^{-1} = g(T_+^i, T_+^j)$.

We observe that

$$\left\{\alpha^{-1}\left[\frac{1-\alpha^{-2}}{z} + z\right]\right\} = q^{-k}\left\{\frac{q^2+q^{-2}-q^{-2k}-1}{q-q^{-1}}\right\} .$$

Therefore, it is enough to prove that

$$\left(P_{\vec{L}(T_+^i,T_+^j)}(q^k, q-q^{-1}) / \left\{q^{-k}\left[\frac{q^2+q^{-2}-q^{-2k}-1}{q-q^{-1}}\right]\right\}^{n-1}\right)_{i,j=1}^{r}$$

is positive semi-definite.

By Lemma 4.2.3 we may omit the factor $q^{-k\mathrm{wr}(\vec{L})} > 0$. A quick computation shows that also the factor

$$\frac{q-q^{-1}}{q^k-q^{-k}}q^{-(k+1)\mathrm{rot}(\vec{L}(T_+^i,T_+^j))}\left[q^{-k}\left(\frac{q^2+q^{-2}-q^{-2k}-1}{q-q^{-1}}\right)\right]^{-n+1}$$

is positive and hence can be neglected. Thus it is enough to show that

$$\Big(\sum_\tau \prod_{x\in V(\vec{L}(T_+^i,T_+^j))} w_+(\tau,x)\; q^{2s(\vec{L}(T_+^i,T_+^j),\tau)}\Big)_{i,j=1}^r$$

is positive semi-definite.

We notice that $s(\vec{L}(T_+^i,T_+^j),\tau) = s(\vec{L}_+(T_+^i),\tau) + s(\vec{L}_+(T_+^i),\tau)$. Therefore we have $q^{s(\vec{L}(T_+^i,T_+^j),\tau)} = q^{s(\vec{L}_+(T_+^i),\tau)}q^{s(\vec{L}_+(T_+^i),\tau)}$.

Each state τ may be decomposed as a triple (τ_0,τ_+,τ_-), where τ_0 is a function on the edges in common between the upper and lower semi-links, τ_+ and τ_- are functions on the remaining edges of $\vec{L}_+(T_+)$ and $\vec{L}_-(T_-)$, respectively.

For any $\tau_0 = (\tau_1,\cdots,\tau_{2n})$, the expression $\sum_{\tau_+}\prod_{x\in V(\vec{L}_+(T_+^i))} w_+(\tau,x)\; q^{2s(\vec{L}_+(T_+^i),\tau)}$ defines the τ_0-th component of a vector $v_{T_+^i}$ in $\mathcal{H} = \mathbb{C}^{k^{2n}}$, i.e the component corresponding to $e_{\tau_1}\otimes\cdots\otimes e_{\tau_{2n}}$. Thus, considering the vectors $v_{T_+^i}$, $i=1,\ldots,r$, it is not difficult to check that , for every $i,j=1,\ldots,r$,

$$\begin{aligned}
&\sum_\tau \prod_{x\in V(\vec{L}(T_+^i,T_+^j))} w_+(\tau,x)\; q^{2s(\vec{L}(T_+^i,T_+^j),\tau)}\\
&\quad= \sum_{\tau_0}\left(\sum_{\tau_+}\prod_{x\in V(\vec{L}(T_+^i))} w_+(\tau,x)\; q^{2s(\vec{L}_+(T_+^i),\tau)}\right)\left(\sum_{\tau_-}\prod_{x\in V(\vec{L}(T_+^j))} w_+(\tau,x)\; q^{2s(\vec{L}_+(T_+^j),\tau)}\right)\\
&\quad= \langle v_{T_+^i}, v_{T_+^j}\rangle.
\end{aligned}$$

It follows that the matrix $\left(P_{\vec{L}(T_+^i,T_+^j)}/\left\{q^{-k}\left[\frac{q^2+q^{-2}-q^{-2k}-1}{q-q^{-1}}\right]\right\}^{n-1}\right)_{i,j=1}^r$ is positive semi-definite for any r, i.e. the function $P_g(q^k,q-q^{-1})$ is of positive type. □

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